Global dimensions of endomorphism algebras of generator-cogenerators over $m$-replicated algebras

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Motivation

- The endomorphism algebras of generator-cogenerators have attracted a lot of interest, and these are just the artin algebras of dominant dimension at least two.
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- The smallest value of the global dimensions of the endomorphism algebras of generator-cogenerators was defined to be the representation dimension by M. Auslander.
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- The smallest value of the global dimensions of the endomorphism algebras of generator-cogenerators was defined to be the representation dimension by M.Auslander.

- In particular, M.Auslander proved that an artin algebra is representation-finite if and only if its representation dimension is at most two.
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- These motivate the investigation on the possibilities for the global dimensions of the endomorphism algebras of generator-cogenerators.

- In general, it is not easy to compute the global dimension of $\text{End}(M)$ whenever $M$ is a generator-cogenerator. Hence constructions of generator-cogenerators with a fixed global dimension have an independent interest.
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- We follow the idea of V.Dlab and C.M.Ringel and investigate the possible values for the global dimensions of the endomorphism algebras of generator-cogenerators for $m$-replicated algebra.
Notations

- Let \( \Lambda \) be an artin algebra.
  - \( \text{gl.dim } \Lambda \) is the global dimension of \( \Lambda \).
  - \( \tau_\Lambda \) is the Auslander-Reiten translation of \( \Lambda \).
- Let \( M \) be a \( \Lambda \)-module.
  - \( \text{pd } M \) is the projective dimension of \( M \).
  - \( \Omega_{\Lambda}^{-k}M \) is the \( k^{\text{th}} \) cosyzygy of \( M \).
Notations

- Let $\Lambda$ be an artin algebra.
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- Let $M$ be a $\Lambda$-module.
  - $\text{pd } M$ is the projective dimension of $M$.
  - $\Omega_\Lambda^{-k} M$ is the $k^{th}$ cosyzygy of $M$.

$M$ is called a generator-cogenerator if all indecomposable projective modules and indecomposable injective modules are in $\text{add } M$.

- Let $M$ be a generator-cogenerator.
  - If $\text{gl.dim End}_\Lambda(M) = d$, then $M$ is also called a generator-cogenerator with global dimension $d$. 

Return
Notations

- From now on, let $A$ be a hereditary algebra.

\[
A^{(m)} = \begin{pmatrix}
A_0 & 0 \\
Q_1 & A_1 \\
Q_2 & A_2 \\
& & \ddots & \ddots \\
& & & 0 & Q_m & A_m
\end{pmatrix}.
\]

is the $m$-replicated algebra of $A$, where $A_i = A$ and $Q_i = DA$, $D$ is the standard duality between $\text{mod } A$ and $\text{mod } A^{op}$, multiplication is induced from the canonical isomorphisms $A \otimes_A DA \cong DA \cong DA \otimes_A A$ and the zero morphism $DA \otimes_A DA \rightarrow 0$. 

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Let $\Sigma_k = \Omega_{A'}^{-k} \Sigma_0 = \{ \Omega_{A'}^{-k} X \mid X \in \Sigma_0 \}$ for $k \geq 0$. $\Sigma_0$ is the set of all non-isomorphic indecomposable projective $A$-modules.
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- Let $U_k$ be the direct sum of all the indecomposable modules in $\Sigma_k \cap \text{ind } A^{(m)}$ for $k \geq 0$. 
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\Sigma_0$ is the set of all non-isomorphic indecomposable projective $A$-modules.

- Let $U_k$ be the direct sum of all the indecomposable modules in $\Sigma_k \cap \text{ind } A^{(m)}$ for $k \geq 0$.

- Let $P$ be the direct sum of all indecomposable projective-injective $A^{(m)}$-modules.
Main results

- **Theorem 1.** Let $d$ be an integer with $d \geq 2$ and $A$ be a representation finite hereditary Artin algebra. There exists an $A^{(m)}$-module $M$ which is a generator-cogenerator with global dimension $d$ if and only if there exists a $\tau_{A^{(m)}}$-orbit of cardinality at least $d$. 
Main results

- **Theorem 1.** Let $d$ be an integer with $d \geq 2$ and $A$ be a representation finite hereditary Artin algebra. There exists an $A^{(m)}$-module $M$ which is a generator-cogenerator with global dimension $d$ if and only if there exists a $\tau^{A^{(m)}}$-orbit of cardinality at least $d$.

- **Remark.** In this finite type case, let $s$ be the maximal length of all $\tau^{A^{(m)}}$-orbits. Then for any $d$ with $2 \leq d \leq s$, one can find a generator-cogenerator $A^{(m)}$-module $M$ such that $	ext{gl.dim } \text{End}_{A^{(m)}}(M) = d$. 
Generator-cogenerators with dimension $i + 2$

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**Proposition 1.** Let $E_i = A \oplus DA_m \oplus P \oplus \bigoplus_{k=i}^{t-1} U_k$ for $1 \leq i \leq t - 1$. Then $E_i$ is a generator-cogenerator $A^{(m)}$-module and $\text{gl.dim } \text{End}_{A^{(m)}}(E_i) = i + 2$. 
Generator-cogenerators with dimension $i + 2$

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**Corollary 2.** Let $d$ be an integer with $3 \leq d \leq t + 1$. Then there exists a generator-cogenerator $M$ in $\text{mod } A^{(m)}$ with global dimension $d$. In particular, the representation dimension of $A^{(m)}$ is at most $3$. 
Generator-cogenerators with dimension $d$

**Proposition 3.** Let $d$ be an integer with $d \geq 2m + 3$. Let $Z$ be an indecomposable non-injective $A^{(m)}$-module such that $\tau^{d-(2m+2)}Z$ is a simple and projective $A$-module. Let

$$0 \rightarrow \tau Z \rightarrow \bigoplus Y_j \rightarrow Z \rightarrow 0$$

be the Auslander-Reiten sequence ending in $Z$, with indecomposable modules $Y_j$. Let

$$M = A \oplus DA_m \oplus \bigoplus_{i=0}^{d-(2m+3)} (\bigoplus \tau^i Y_j) \oplus P.$$ 

Then $M$ is a generator-cogenerator $A^{(m)}$-module and $\text{gl.dim } \text{End}_{A^{(m)}}(M) = d$. 

▶ Return
Generator-cogenerators with dimension $\infty$

- **Proposition 4.** Let $N$ be an indecomposable $A$-module whose endomorphism algebra is a division ring and such that there is a non-split sequence $0 \to N \xrightarrow{u} N' \xrightarrow{v} N \to 0$. Let $M = A \oplus DA_m \oplus P \oplus N'$. Then $M$ is a generator-cogenerator in $\text{mod } A^{(m)}$ and $\text{gl.dim } \text{End}_{A^{(m)}}(M) = \infty$. 
Main results

- According to Proposition 1, Proposition 3 and Proposition 4, we get the following theorem.
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According to Proposition 1, Proposition 3 and Proposition 4, we get the following theorem.

Theorem 2. Let $A$ be a representation infinite hereditary Artin algebra and $d$ be either an integer with $d \geq 3$ or else the symbol $\infty$. Then there exists a generator-cogenerator $A^{(m)}$-module $M$ with $\text{gl.dim End}_{A^{(m)}}(M) = d$. 
Main results

- According to Proposition 1, Proposition 3 and Proposition 4, we get the following theorem.

**Theorem 2.** Let $A$ be a representation infinite hereditary Artin algebra and $d$ be either an integer with $d \geq 3$ or else the symbol $\infty$. Then there exists a generator-cogenerator $A^{(m)}$-module $M$ with \( \text{gl.dim } \text{End}_{A^{(m)}}(M) = d \).

**Remark.** In this infinite type case, for any $d$ with $3 \leq d \leq \infty$, we explicitly construct a generator-cogenerator $A^{(m)}$-module $M$ with \( \text{gl.dim } \text{End}_{A^{(m)}}(M) = d \).
Main References


THANK YOU!