Some Studies on Radiation Hydrodynamics Models

Feng Xie

Department of Mathematics,
Shanghai Jiao Tong University

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Outline

1 Introduction
   - General Radiation Hydrodynamics System
   - “Baby Model” and Related Model in RHD
   - Euler-Elliptic Coupled Systems in RHD

2 Main Results
   - Formulation of Problem
   - Main Results in This Talk

3 Sketch of Proof
   - Local Existence
   - Uniform A Priori Estimates
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(1) Nonrelativistic equations of hydrodynamics for a compressible, ideal fluid (with Radiation):

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\frac{\partial}{\partial t} \left( \rho u + \frac{1}{c^2} F_r \right) + \nabla P_m + \nabla \cdot (\rho u \otimes u + P_r) &= 0, \\
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + E_m + E_r \right) + \nabla \left[ \left( \frac{1}{2} \rho u^2 + E_m + P_m \right) u + F_r \right] &= 0,
\end{align*}
\]

\(\rho\) density, \(u\) velocity vector, \(P_m\) pressure, \(E_m\) inertial energy;(Fluid Part)

\(E_m\) radiation energy, \(P_r\) radiation pressure, \(F_r\) radiation flux. (Radiation Part)
where

\[ E_r = \frac{1}{c} \int_0^\infty dv \int_{4\pi} d\Omega I(\nu, \Omega), \]

\[ F_r = \int_0^\infty dv \int_{4\pi} d\Omega \Omega I(\nu, \Omega), \]

\[ P_r = \frac{1}{c} \int_0^\infty dv \int_{4\pi} d\Omega \Omega \Omega I(\nu, \Omega), \]
Radiation Hydrodynamics Models I

The specific intensity of radiation $I(\nu, \Omega)$ is governed by the following equation

\[
\frac{1}{c} \frac{\partial I(\nu, \Omega)}{\partial t} + \Omega \cdot \nabla I(\nu, \Omega) = S(\nu) - \sigma_a(\nu)I(\nu, \Omega)
\]

\[
+ \int_0^\infty d\nu' \int_4^{\pi} d\Omega' \left[ \frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \Omega' \Omega)I(\nu', \Omega) - \sigma_s(\nu \to \nu', \Omega\Omega')I(\nu, \Omega) \right]
\]

$\sigma_a$ absorbing coefficient, $\sigma_s$ scattering coefficient, $S$ Planck function.

Studies on general radiation hydrodynamics model:


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Radiation Hydrodynamics Models I

- Simplification in Fluid Part:
  Euler System $\rightarrow$ Burgers Equation.

- Simplification in Radiation Part:
  Equation of Transfer $\rightarrow$ Elliptic Type Equation (Diffusion Approximation).

**Remark**

In general, when we apply for diffusion approximation, the equation of transfer is changed into diffusion equation. In fact, one can carry out further simplification by ignore the time derivative term due to the factor $\frac{1}{c}$ is small enough.
(1) Hamer Model (Baby Model)

\[
\begin{align*}
    u_t + \left( \frac{u^2}{2} \right)_x + q_x &= 0, \\
    -q_{xx} + q + u_x &= 0,
\end{align*}
\]

(3)

It is a “simplest” Radiation Hydrodynamics Model. Which is Hyperbolic-Elliptic coupled system

Known Results about "Baby Model" I

Existence and Blowup Phenomena of Solution:


Global existence \((u'_0(x) > u^*)\), Blowup \((u'_0(x) < u^*)\), \(u^*, u_*\) are negative constants. Existence of weak solutions;

\[
q_x = u - Ku
\]

where

\[
(Kf)(x) = \left(-\frac{\partial^2}{\partial x^2} + 1\right)^{-1} = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|x-y|} f(y) dy.
\]
Known Results about "Baby Model" I

Stability of Shock Profile:

- ...

Known Results about "Baby Model" I

Stability of Rarefaction Wave:

- W. Gao, L. Ruan, C. Zhu Jour. Diff. Equations 2008. Stability of rarefaction waves (Multi-D);
- W. Wang, R. Duan, Y. Liu, S. Kawashima,...
Hyperbolic-Hyperbolic(Parabolic) Relaxation limit Problems: Applying the hyperbolic(parabolic) scaling to the "Baby Model" yields the following two singular limit problems

\[ u_t + \left(\frac{u^2}{2}\right)_x + q_x = 0, \quad -\varepsilon^2 q_{xx} + q + \varepsilon u_x = 0, \]

and

\[ u_t + \left(\frac{u^2}{2}\right)_x + q_x = 0, \quad -\varepsilon q_{xx} + q + u_x = 0, \]

Radiation Hydrodynamics Models I

We consider the Burgers’ equation coupled with steady equation of transfer in plane geometry:

\[
\begin{aligned}
  u_t + uu_x &= - \int_{-1}^{1} S d\mu, \\
  \mu I_x &= S, \quad S = S_a + S_c, \\
  u(x, t = 0) &= u_0(x),
\end{aligned}
\]  

(4)

with

\[
S_a = \sigma_a |\mu| (u - I), \quad S_c = \sigma_s (\bar{I} - I), \quad \bar{I} = \frac{1}{2} \int_{-1}^{1} Id\mu.
\]
Remark

Since the light speed $c$ is large enough, the term with the factor $\frac{1}{c}$ can be ignored, Consequently, we consider the steady equation of transfer.

Remark

We should remark that if we set $\sigma_a = 1$ and $\sigma_s = 0$, the system (4) can be equivalently rewritten as Hamer model(“Baby Model”).
(Rohde and Xie MMAS 2012) Suppose $u_0(x) \in B^1(R)$ and if there exists a point $\xi \in R$ such that

$$u'_0(\xi) \leq \theta_* = \frac{-\sigma_a - \sqrt{\sigma_a^2 + 4\sigma_aka_0}}{2}.$$

Then the Cauchy problem (4) can not admit a global classical solution. Moreover, $u_x(x,t)$ blows up in a finite time

$$\lim_{t \to t_*} u_x(x,t) = -\infty$$

for a certain finite time $t_* > 0$. 
Theorem

Suppose \( u_0(x) \in B^1(R) \) and satisfies

\[
|b_0 - a_0| \leq \frac{\sigma_a^2}{2\sigma_a + 4\sigma_s}
\]

and

\[
u'_0(x) > \frac{-\sigma_a - \sqrt{\sigma_a^2 - 4\delta_0}}{2}.
\]

Then the Cauchy problem (4) admits a global classical solution. Moreover, the solution satisfies

\[
\min\left\{\frac{-\sigma_a + \sqrt{\sigma_a^2 - 4\delta_0}}{2}, a_1\right\} \leq u_x(x,t) \leq b_1,
\]
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(2) The Hamer model (Baby model) is the third order approximation of the following system (In Lagrangian Coordinates).

\[
\begin{align*}
 v_t - u_x &= 0, \\
 u_t + p_x &= 0, \\
 \left(e + \frac{u^2}{2}\right)_t + (pu)_x + q_x &= 0, \\
 -\left(\frac{q_x}{v}\right)_x + avq + b(\theta^4)_x &= 0,
\end{align*}
\]

(5)

\( v > 0 \) specific volume, \( u \) velocity, \( \theta > 0 \) absolute temperature, \( e > 0 \) internal energy, \( p \) pressure and \( q \) radiation flux.
Known Results about Model 2

Existence of Solution:

- S.Kawashima, Y.Nikkuni, S.Nishibata,

Global existence of solution and large time behavior of solution to the small perturbation problem near constant states

\[ \| v_0(x) - v^*, u_0(x) - u^*, \theta_0(x) - \theta^* \|_{H^2(\mathbb{R})} \text{ small enough}. \]
Known Results about Model 2

Existence of Shock Profile:

  Global existence of shock profile to system (5);

  Global existence of shock profile to general hyperbolic-elliptic coupled system.
**Known Results about Model 2**

**Stability of Shock Profile:**

Known Results about Model 2

Stability of Rarefaction Wave:
  Stability of viscous rarefaction wave of system (5)

Stability of viscous contact discontinuity wave:
  Stability of viscous contact wave of system (5);
- Rohde C., Xie F. M3AS 2012.
  Convergence rate to the viscous contact wave.

These are the main parts of our talk.
Two Singular Problems for (5):

Singular Limit Problem I

Let the Bouguer number became infinite and Boltzmann number became infinitesimal, with their product kept constant (Or applying hyperbolic-parabolic scaling to model (5)).

\[
\begin{aligned}
&v_t - u_x = 0, \\
&u_t + P_x = 0, \\
&(e + \frac{u^2}{2})_t + (Pu)_x + q_x = 0, \\
&-\varepsilon\left(\frac{q_x}{v}\right)_x + vq + (\theta^4)_x = 0,
\end{aligned}
\]  

(6)
Kawashima S., Nishibata S.
Solution to (6) converges to the solution to corresponding hyperbolic-parabolic system, as $\varepsilon$ tends to zero. Moreover, convergence rates are given.
Known Results about Model 2 I

Singular Limit problem II:
Let the Bouguer number became infinite and kept Boltzmann number constant (Or applying hyperbolic-parabolic scaling to model (5)).

\[
\begin{align*}
    v_t - u_x &= 0, \\
    u_t + P_x &= 0, \\
    \left( e + \frac{u^2}{2} \right)_t + (Pu)_x + q_x &= 0, \\
    -\varepsilon^2 \left( \frac{q_x}{v} \right)_x + v q + \varepsilon (\theta^4)_x &= 0,
\end{align*}
\]

(7)

Where \( \varepsilon \) is the reciprocal of Bouguer number.
For RHD Model (7)

- Wang J., Xie F. SIAM Math. Anal. 2011. Solution to (7) converges to the contact wave to corresponding Euler system in any finite time, as $\varepsilon$ tends to zero. Moreover, convergence rates are given in terms of $\varepsilon$.

- Wang, Xie and Rohde CPAA 2012. Solution to (7) converges to the composition wave of two rarefaction waves and one contact wave to corresponding Euler system in any finite time, as $\varepsilon$ tends to zero. Moreover, convergence rates are given in terms of $\varepsilon$. 
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First we consider the following hyperbolic-hyperbolic relaxation limit problem

\[
\begin{aligned}
    v_t - u_x &= 0, \\
    u_t + P_x &= 0, \\
    (e + \frac{u^2}{2})_t + (Pu)_x + q_x &= 0, \\
    -\varepsilon^2 (\frac{q_x}{v})_x + vq + \varepsilon (\theta^4)_x &= 0,
\end{aligned}
\]
Here we focus on the ideal polytropic gas, that is, the state equations take the form

\[ p = R \frac{\theta}{v} = A v^{-\gamma} e^{\frac{\gamma-1}{R} s}, \quad e = \frac{R}{\gamma-1} \theta, \]

where \( s \) is the entropy, \( \gamma > 1 \) is the adiabatic exponent and \( A, R \) are two positive constants.
When $\varepsilon = 0$, the system (8) is reduced into the classical 1D full Euler system

$$
\begin{align*}
    v_t - u_x &= 0, \\
    u_t + P_x &= 0, \\
    (e + \frac{u^2}{2})_t + (P u)_x &= 0,
\end{align*}
$$

(9)
The Euler system (9) admits the contact wave

$$\begin{cases} (\bar{V}, \bar{U}, \bar{\Theta})(x,t) = (v_-, u_-, \theta_-), & x < 0, \quad t > 0, \\ (v_+, u_+, \theta_+), & x > 0, \quad t > 0, \end{cases} \quad (10)$$

provided that

$$u_- = u_+, \quad P_- \triangleq \frac{R\theta_-}{v_-} = \frac{R\theta_+}{v_+} \triangleq P_+. \quad (11)$$
Theorem

Let \((\bar{V}, \bar{U}, \bar{\Theta})\) be the contact wave defined in (10) for the Euler system (9). Then for any time \(T > 0\) and small \(\sigma > 0\), there exists a constant \(\varepsilon_0 > 0\), such that for each \(\varepsilon \in (0, \varepsilon_0)\) the system (8) admits a unique smooth solution \((v, u, \theta, q)\). Moreover,

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}} \| (v, u, \theta)(x, t) - (\bar{V}, \bar{U}, \bar{\Theta})(x, t) \|^2 dx \leq C\varepsilon^{1/2}, \tag{12}
\]

\[
\sup_{0 \leq t \leq T, |x| > \sigma} \| (v, u, \theta)(x, t) - (\bar{V}, \bar{U}, \bar{\Theta})(x, t) \| \leq C\varepsilon^{1/4}, \tag{13}
\]

where \(C\) is a positive constant independent of \(\varepsilon\).
Construction of Approximation Solution \((V, U, \Theta, Q)\) to the system \((8)\)

Approximation I:

\[\tilde{P} \equiv \frac{R\Theta}{V} \approx P_+.\]  \hspace{1cm} (14)

Approximation II:

\[VQ \approx -\varepsilon (\Theta^4)_x.\]  \hspace{1cm} (15)
Then the leading part of the energy equation $(8)_3$ is

$$\frac{R}{\gamma - 1} \Theta_t + P_+ U_x = \epsilon \left( \frac{4 \Theta^3 \Theta_x}{V} \right)_x. \quad (16)$$

The equation (16), (14) and $(8)_1$ lead to the nonlinear diffusion equation

$$\Theta_t = \epsilon \frac{\gamma - 1}{R} \left( \frac{4 P_+}{R} \Theta^2 \Theta_x \right)_x, \quad \Theta(\pm \infty, t) = \theta_\pm, \quad (17)$$
which has a unique self-similarity solution

\[ \Theta(x, t) = \Theta(\xi), \xi = \frac{x}{\sqrt{1+t}}. \]

Furthermore, \( \Theta(\xi) \) is a monotone increasing function if \( \theta_+ > \theta_- \) and a monotone decreasing function if \( \theta_+ < \theta_- \). Let \( \delta = |\theta_+ - \theta_-| \), then \( \Theta(\xi) \) satisfies

\[
| (\varepsilon (1 + t))^{1/2} \partial_x^l \Theta | + | \Theta - \theta_\pm | \leq c_1 \delta e^{-c_2 x^2/\varepsilon (1+t)}, \quad \text{as } |x| \to \infty, \quad l \geq 1,
\]

where \( c_1, c_2 \) are two positive constants depending only on \( \theta_+ \) and \( \delta \).
Once $\Theta$ is determined, the “viscous contact wave” \((V, U, \Theta, Q)(x, t)\) is defined as

\[
V = \frac{R}{P_+} \Theta, \quad U = u_- + \varepsilon \frac{(\gamma - 1)}{R \gamma} 4 \Theta^2 \Theta_x, \quad \Theta = \Theta, \quad Q = -\varepsilon \frac{4P_+}{R} \Theta^2 \Theta_x.
\]

(19)

It is straightforward to check that the “viscous contact wave” \((V, U, \Theta, Q)(x, t)\) satisfies

\[
\|V - \bar{V}, U - \bar{U}, \Theta - \bar{\Theta}\|_{L^p(\mathbb{R})} = O(\varepsilon^{1/(2p)}) (1 + t)^{1/(2p)}, \quad p \geq 1,
\]

(20)
In this way, we only need to show the difference between the viscous contact wave (approximation solution) and the solution to (8) is bounded by $\varepsilon^a$, $a$ is some positive constant. It is nature for us to consider the stability of viscous contact wave for the system (8).

Using the following hyperbolic scalings

$$x = \frac{x}{\varepsilon}, \quad t = \frac{t}{\varepsilon},$$

(21)
Then the system (8) is changed into the following system

\[
\begin{cases}
  v_t - u_x = 0, \\
  u_t + p_x = 0, \\
  (e + \frac{u^2}{2})_t + (pu)_x + q_x = 0, \\
  -(\frac{q_x}{v})_x + avq + b(\theta^4)_x = 0,
\end{cases}
\]

(22)

with initial data and far field states

\[
\begin{cases}
  (v, u, \theta)(x, 0) = (v_0(x), u_0(x), \theta_0(x)), & x \in \mathbb{R}, \\
  (v, u, \theta, q)(\pm \infty, t) = (v_\pm, u_\pm, \theta_\pm, 0), & t > 0.
\end{cases}
\]

(23)
Construction of viscous contact wave

CONSTRUCTION OF VISCOUS CONTACT WAVE:

\[ p^{cd} \equiv \frac{R \Theta^{cd}}{V^{cd}} = p_+ , \]

and the radiation flux \( Q^{cd} \) is expected to act as a diffusion term

\[ Q^{cd} = -\frac{b}{aV^{cd}} ((\Theta^{cd})^4)_x . \]

Then the leading order of the energy equation (5)_3 can be written as

\[ \frac{R}{\gamma - 1} \Theta^{cd}_t + p_+ U^{cd}_x = \left( \frac{4b}{a} \frac{(\Theta^{cd})^3 \Theta^{cd}_x}{V^{cd}} \right)_x . \]
Then we have the following nonlinear diffusion equation

\[
\Theta_{t}^{cd} = \frac{\gamma - 1}{R\gamma} \left( \frac{4bp+}{aR} \left( \Theta^{cd} \right)^2 \Theta^{cd}_{x} \right)_x, \quad \Theta^{cd}(\pm\infty, t) = \theta_{\pm}, \tag{25}
\]

which has a unique self-similarity solution \( \Theta^{cd}(x, t) = \Theta^{cd}(\xi), \xi = \frac{x}{\sqrt{1+t}} \). Furthermore, \( \Theta^{cd}(\xi) \) is a monotone function. On the other hand, there exists a positive constant \( \bar{\delta} \), such that for \( \delta_2 = \left| \theta_+ - \theta_- \right| \leq \bar{\delta}, \Theta^{cd}(\xi) \) satisfies

\[
(1 + t)^{l/2} |\partial_x^l \Theta^{cd}| + |\Theta^{cd} - \theta_{\pm}| \leq c_1 \delta_2 e^{-c_2 x^2/(1+t)}, \quad |x| \rightarrow \infty, \quad l \geq 1, \tag{26}
\]
the “viscous contact wave” profile \((V^{cd}, U^{cd}, \Theta^{cd}, Q^{cd})(x, t)\) is then defined as

\[
V^{cd} = \frac{R}{p_+} \Theta^{cd}, \quad U^{cd} = u_- + \frac{(\gamma - 1)}{R\gamma} \frac{4b}{a} (\Theta^{cd})^2 \Theta^{cd}_x,
\]

\[
\Theta^{cd} = \Theta^{cd}, \quad Q^{cd} = -\frac{4bp_+}{aR} (\Theta^{cd})^2 \Theta^{cd}_x.
\]  
(27)

It is straightforward to check that the “viscous contact wave” \((V^{cd}, U^{cd}, \Theta^{cd}, Q^{cd})(x, t)\) solves the system (5) time asymptotically, that is

\[
\begin{cases}
V^{cd}_t - U^{cd}_x = 0, \\
U^{cd}_t + \left(\frac{R\Theta^{cd}}{V^{cd}}\right)_x = R_1, \\
(e(V^{cd}, \Theta^{cd}) + \frac{(U^{cd})^2}{2})_t + (p(V^{cd}, \Theta^{cd}) U^{cd})_x + Q^{cd}_x = R_2, \\
-(\frac{Q^{cd}}{V^{cd}})_x + aV^{cd} Q^{cd} + b((\Theta^{cd})^4)_x = R_3,
\end{cases}
\]  
(28)
where

\[ R_1 = U^{cd}_t = \left( \frac{\gamma - 1}{R\gamma} \right) \frac{4b}{a} \left( \Theta^{cd} \right)^2 \frac{\Theta^{cd}_x}{x} \]

\[ = O(\delta_2)(1 + t)^{-3/2} e^{-c_2x^2/(1+t)}, \quad \text{as} \quad |x| \to \infty, \]

\[ R_2 = U^{cd} U^{cd}_t = (u_- + \left( \frac{\gamma - 1}{R\gamma} \right) \frac{4b}{a} \left( \Theta^{cd} \right)^2 \frac{\Theta^{cd}_x}{x} ) \left( \frac{\gamma - 1}{R\gamma} \right) \frac{4b}{a} \left( \Theta^{cd} \right)^2 \frac{\Theta^{cd}_x}{x} \]

\[ = O(\delta_2)(1 + t)^{-3/2} e^{-c_2x^2/(1+t)}, \quad \text{as} \quad |x| \to \infty, \]

and

\[ R_3 = - \left( \frac{Q^{cd}_x}{V^{cd}} \right)_x = \frac{4b(p_+)^2}{aR^2} 2 \left( \left( \Theta^{cd} \right)^2 + \Theta^{cd} \frac{\Theta^{cd}_x}{x} \right)_x \]

\[ = O(\delta_2)(1 + t)^{-3/2} e^{-c_2x^2/(1+t)}, \quad \text{as} \quad |x| \to \infty. \]
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Thus, we are ready to state our main results.

**Theorem**

Suppose $|v_+ - v_-| + |u_+ - u_-| + |	heta_+ - \theta_-| \leq \delta_1$ and $\delta_1$ is small. Let $(V, U, \Theta, Q)$ be defined as in (28). Then there exist suitably small positive constants $\delta_0 \leq \delta_1$ and $\sigma_0$, such that if

$$
\delta = |\theta_+ - \theta_-| < \delta_0 \quad \text{and the initial data } (v_0, u_0, \theta_0) \text{ satisfies}
$$

$$
\| (v_0(\cdot) - V(\cdot, 0), u_0(\cdot) - U(\cdot, 0), \theta_0(\cdot) - \Theta(\cdot, 0)) \|_2 \leq \sigma_0,
$$

then the Cauchy problem (22)-(23) admits a unique global solution $(v, u, \theta, q)$ satisfying $(v - V, u - U, \theta - \Theta, q - Q) \in E_{[0, \infty]}$. Moreover,

$$
\limsup_{t \to \infty} \sup_{x \in \mathbb{R}} |(v - V, u - U, \theta - \Theta, q - Q)(x, t)| = 0.
$$
Set

\[ \phi(x, t) = v - V, \quad \psi(x, t) = u - U, \quad \zeta(x, t) = \theta - \Theta, \quad \omega(x, t) = q - Q, \]

\[ \Phi(x, t) = \int_{-\infty}^{x} \phi(y, t) dy, \quad \Psi(x, t) = \int_{-\infty}^{x} \psi(y, t) dy, \]

\[ \bar{W} = \int_{-\infty}^{x} \left( e + \frac{u^2}{2} - e(V, \Theta) - \frac{U^2}{2} \right)(y, t) dy, \]

\[ \partial_x Z = (V + \partial_x \Phi) \omega + \phi Q. \]
Theorem

Suppose the conditions in Theorem 1 are satisfied and
\[
\| (\Phi, \Psi, \bar{W})(\cdot, t = 0) \|_{H^3(\mathbb{R})} \leq \varepsilon,
\]
then the system (5) admits a unique global solution
\((v, u, \theta, q)(x, t)\) satisfy

\[
(\Phi, \Psi, \bar{W}, \omega) \in C(0, +\infty; H^3), \quad (\phi, \psi, \zeta, \omega, \omega_x) \in L^2(0, +\infty; H^2).
\]

Furthermore, the solution satisfies
\[
\| (v - V, u - U, \theta - \Theta, q - Q) \|_{L^\infty(\mathbb{R})} \leq C(\varepsilon + \delta_0^{1/2})(1 + t)^{-1/4}.
\]
Remark

In the above theorem, we assume the initial perturbation is zero mass perturbation. In fact, we do not require the zero mass perturbation assumption. When the initial perturbation is not zero mass perturbation, we can introduce the two diffusion waves explicitly which are uniquely determined by the initial perturbation. Then we can show the combination of the viscous contact wave and two diffusion waves is stable and give the same decay rate.
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Some Studies on RHD Models
We can rewrite the system (5) in term of \((\Phi, \Psi, \tilde{W}, Z)\)

\[
\begin{align*}
\Phi_t - \Psi_x &= -\tilde{R}_0, \quad \Phi(0) = \Phi_0, \\
\Psi_t - \frac{R\tilde{\theta}\Phi_x}{\tilde{v}(\tilde{v} + \Phi_x)} + \frac{R\zeta}{\tilde{v} + \Phi_x} &= -\tilde{R}_1, \quad \Psi(0) = \Psi_0, \\
\frac{R}{\gamma - 1} W_t + \frac{R(\tilde{\theta} + \zeta)\Psi_x}{\tilde{v} + \Phi_x} + \omega &= -\tilde{u}_t\Psi + \tilde{u}\tilde{R}_1 - \tilde{R}_2, \quad W(0) = W_0, \\
-\left(\frac{\tilde{q}_x + \omega_x}{\tilde{v} + \Phi_x}\right)_x + (\tilde{v} + \Phi_x)\omega &= -\tilde{q}\Phi_x - ((\tilde{\theta} + \zeta)^4 - \tilde{\theta}^4)_x,
\end{align*}
\]

(29)

where

\[
\zeta = W_x - \frac{\gamma - 1}{R}\left(\frac{1}{2}\Psi_x^2 - \tilde{u}_x\Psi\right).
\]

(30)
First, we rewrite the first three equations in system (29) as follows.

\[
\begin{align*}
\Phi_t - \Psi_x &= F_1, \\
\Psi_t - \frac{R\tilde{\theta}}{\tilde{v}^2} \Phi_x + \frac{R}{\tilde{v}} W_x &= F_2, \\
\frac{R}{\gamma - 1} W_t + \frac{R\tilde{\theta}}{\tilde{v}} \Psi_x &= F_3,
\end{align*}
\]

(31)

where

\[
\begin{align*}
F_1 &= -\tilde{R}_0, \\
F_2 &= -\tilde{R}_1 + \frac{R\tilde{\theta}\Phi_x}{\tilde{v}(\tilde{v} + \Phi_x)} - \frac{R\tilde{\theta}\Phi_x}{\tilde{v}^2} + \frac{RW_x}{\tilde{v}} - \frac{R\zeta}{\tilde{v} + \Phi_x}, \\
F_3 &= -\omega - \tilde{u}_t \Psi + \tilde{u}\tilde{R}_1 - \tilde{R}_2 + \frac{R\tilde{\theta}\Psi_x}{\tilde{v}} - \frac{R(\tilde{\theta} + \zeta)\Psi_x}{\tilde{v} + \Phi_x}.
\end{align*}
\]

(32)
We put $U = (\Phi, \Psi, \frac{R}{\gamma - 1} W)^T$ and $\tilde{U} = (\tilde{v}, \tilde{u}, \tilde{\theta})^T$, where the superscript $T$ denotes the transposed. Then the equations in (31) are written in the vector form as

$$U_t + A(\tilde{U}) U_x = F.$$  \hspace{1cm} (33)

Here

$$A(\tilde{U}) = \begin{pmatrix}
0 & -1 & 0 \\
-\frac{R \tilde{\theta}}{\tilde{v}^2} & 0 & \frac{\gamma - 1}{\tilde{v}} \\
0 & \frac{R \tilde{\theta}}{\tilde{v}} & 0
\end{pmatrix}$$

and $F = (F_1, F_2, F_3)^T (\tilde{U}, \tilde{U}_{t/x}, \tilde{R}_{0/1/2}, U, U_x, \omega)$. 
We differentiate the system (33) with respect to $x$, then the resulting system is written as

$$U_{xt} + B(\tilde{U}, U, U_x) U_{xx} = G,$$

(34)

where

$$B(\tilde{U}, U, U_x) = \begin{pmatrix}
0 & -1 & 0 \\
-R(\tilde{\theta} + \zeta) & -(\gamma - 1)\Psi_x & \frac{\gamma - 1}{(\tilde{\nu} + \Phi_x)} \\
-R(\tilde{\theta} + \zeta) \Psi_x & \frac{R(\tilde{\theta} + \zeta)}{(\tilde{\nu} + \Phi_x)^2} & \frac{R(\tilde{\theta} + \zeta)}{(\tilde{\nu} + \Phi_x)} - (\gamma - 1)\Psi_x^2 \\
-R(\tilde{\theta} + \zeta) \Psi_x & \frac{R(\tilde{\theta} + \zeta)}{(\tilde{\nu} + \Phi_x)^2} & \frac{R(\tilde{\theta} + \zeta)}{(\tilde{\nu} + \Phi_x)} - (\gamma - 1)\Psi_x^2
\end{pmatrix}$$

(35)
and \( G = (G_1, G_2, G_3)^T (\tilde{U}, \tilde{U}_{xt/xx/tx}, \tilde{R}_{0x/1x/2x}, U, U_x, \omega_x) \) with

\[
G_1 = -\tilde{R}_{0x},
\]

\[
G_2 = -\tilde{R}_{1x} + R \frac{\tilde{\theta}_x \Phi_x}{(\tilde{\nu} + \Phi_x) \tilde{\nu}} + R\tilde{\nu} \left( \frac{(\tilde{\theta} + \zeta)}{(\tilde{\nu} + \Phi_x)^2} - \frac{\tilde{\theta}}{\tilde{\nu}^2} \right) - \frac{\gamma - 1}{(\tilde{\nu} + \Phi_x)} (\tilde{u}_x \Psi)_x,
\]

\[
G_3 = (-\omega - \tilde{u}_t \Psi + \tilde{u}\tilde{R}_1 - \tilde{R}_2)_x - \frac{(\gamma - 1) \Psi_x}{(\tilde{\nu} + \Phi_x)} (\tilde{u}_x \Psi)_x - \frac{R\tilde{\theta}_x \Psi_x}{(\tilde{\nu} + \Phi_x)} + \frac{R(\tilde{\theta} + \zeta) \tilde{\nu}_x \Psi_x}{(\tilde{\nu} + \Phi_x)^2}.
\]
Set \( V = U_x \), then we have

\[
U_t + A(\tilde{U}) U_x = F(\tilde{U}, \tilde{U}_{t/x}, \tilde{R}_{0/1/2}, U, V, \omega), \quad U(0) = U_0,
\]

\[
V_t + B(\tilde{U}, U, V) V_x = G(\tilde{U}, \tilde{U}_{x/t/xx/tx}, \tilde{R}_{0x/1x/2x}, U, V, \omega_x), \quad V(0) = U_{0x}
\]

\[
(36)
\]

\[
- \left( \frac{\omega_x}{\tilde{v} + \Phi_x} \right)_x + (\tilde{v} + \Phi_x) \omega = -\tilde{q}\Phi_x - ((\tilde{\theta} + \zeta)^4 - \tilde{\theta}^4)_x + \left( \frac{\tilde{q}_x}{\tilde{v} + \Phi_x} \right)_x.
\]

We regard system (36) as our problem for new unknown \((U, V, \omega)\), and we want to show the local existence of “small" solution to this problem.
Theorem

Suppose that $U_0 \in H^3(\mathbb{R})$ and $E_0 + \delta$ is suitably small, where $E_0 = \|U_0\|_{H^3(\mathbb{R})}$. Then there exists a positive constant $T_0 > 0$ such that initial value problem (36) has a unique solution $(U, V, \omega)(x, t)$ satisfying

$$(U, V) \in C^0([0, T_0]; H^2) \cap C^1([0, T_0]; H^1), \quad \omega \in C^0([0, T_0]; H^3).$$

Moreover, the solution verifies

$$\|U, V\|_{H^2}^2 + \|\omega\|_{H^3}^2 + \int_0^t \left(\|U_x, V_x\|_{H^1}^2 + \|\omega\|_{H^3}^2\right) d\tau$$

$$\leq C\left(\|U_0, V_0\|_{H^2}^2 + \delta^2\right),$$

for $t \in [0, T_0]$ and $C > 1$. 

Feng Xie (Shanghai Jiao Tong University)  Some Studies on RHD Models
Outline

1. Introduction
   - General Radiation Hydrodynamics System
   - “Baby Model” and Related Model in RHD
   - Euler-Elliptic Coupled Systems in RHD

2. Main Results
   - Formulation of Problem
   - Main Results in This Talk

3. Sketch of Proof
   - Local Existence
   - Uniform A Priori Estimates
Uniform a priori estimates.

(1) $L_t^\infty(L_x^2)$-norm of $(\Phi, \Psi, W)$;
(2) $L_t^2(L_x^2)$-norm of $(\Phi_x, \Psi_x, W_x)$;
(3) $L_t^\infty(L_x^2)$ and $L_t^2(L_x^2)$-norms of $(\phi_x, \psi_x, \zeta_x)$
(4) Estimates of high order derivatives.
THANK YOU!