Spectral Analysis of Growing Graphs
A Quantum Probability Point of View
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5. Asymptotic Spectral Analysis of Growing Regular Graphs
5.1. Main Theme

Growing graphs and spectral distributions

Our Main Theme

The asymptotic behavior of $\mu_n$ as $n \to \infty$. In fact, we will investigate the limit:

$$\lim_{n \to \infty} \mu_n$$
5.2. Simple Example (I) \( P_n \) as \( n \to \infty \)

\[
\text{Spec} (P_n) = \left\{ 2 \cos \frac{k\pi}{n+1} ; 1 \leq k \leq n \right\}
\]

\[
\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{2 \cos \frac{k\pi}{n+1}}
\]

For \( f \in C_b(\mathbb{R}) \) we have

\[
\int_{-\infty}^{+\infty} f(x) \mu_n (dx) = \frac{1}{n} \sum_{k=1}^{n} f \left( 2 \cos \frac{k\pi}{n+1} \right)
\]

\[
\rightarrow \int_{0}^{1} f(2 \cos \pi t) dt
\]

\[
= \int_{-2}^{+2} f(x) \frac{dx}{\pi \sqrt{4-x^2}}.
\]
5.2. Simple Example (II) $K_n$ as $n \to \infty$

$K_n$ as $n \to \infty$

$\text{Spec } (K_n) = \{-1(n-1), n-1(1)\}$

$\mu_n = \frac{1}{n} \delta_{n-1} + \frac{n-1}{n} \delta_{-1}$

Let us see what happens in the limit $\mu_n$ as $n \to \infty$

For $f \in C_b(\mathbb{R})$ we have

$$\int_{-\infty}^{+\infty} f(x) \mu_n(dx) = \frac{1}{n} f(n-1) + \frac{n-1}{n} f(-1)$$

$$\to f(-1) = \int_{-\infty}^{+\infty} f(x) \delta_{-1}(dx) \quad \text{as } n \to \infty$$

This means that $\mu_n \to \delta_{-1}$

Can we accept it? What about the mean values?
5.2. Simple Example (II) $K_n$ as $n \to \infty$

Normalization is a basic idea in probability theory to grasp the limit distribution.

E.g., central limit theorem (CLT) and its variants.

**Definition (normalization)**

For a probability distribution $\mu$ its *normalization* is a probability distribution $\tilde{\mu}$ defined by

$$\int f(x) \tilde{\mu}(dx) = \int f\left(\frac{x - m}{\sigma}\right) \mu(dx),$$

where

$$m = \text{mean}(\mu), \quad \sigma^2 = \text{var}(\mu).$$

Then we have

$$\text{mean}(\tilde{\mu}) = 0, \quad \text{var}(\tilde{\mu}) = 1.$$
5.2. Simple Example (II) $K_n$ as $n \to \infty$

$K_n$ as $n \to \infty$

Spectral distribution (eigenvalue distribution): $\mu_n = \frac{1}{n} \delta_{n-1} + \frac{n-1}{n} \delta_{-1}$

Since $\text{mean}(\mu_n) = 0$ and $\text{var}(\mu_n) = n - 1$, after normalization we have

$$
\int_{-\infty}^{+\infty} f(x) \tilde{\mu}_n(dx) = \frac{1}{n} f\left(\frac{n-1}{\sqrt{n-1}}\right) + \frac{n-1}{n} f\left(\frac{-1}{\sqrt{n-1}}\right)
$$

$\rightarrow f(0) = \int_{-\infty}^{+\infty} f(x) \delta_0(dx)$ as $n \to \infty$.

This means that $\tilde{\mu}_n \to \delta_0$. 

![Normalized Limit of Spectral Distribution](image-url)
5.3. Formulation of Question in General

A difference between $K_n$ and $P_n$ as $n \to \infty$

\[
\mu_{P_n} = \frac{1}{n} \sum_{k=1}^{n} \delta_{2 \cos \frac{k\pi}{n+1}}, \quad \mu_{K_n} = \frac{1}{n} \delta_{n-1} + \frac{n-1}{n} \delta_{-1}
\]

mean value

\[
\text{mean}(\mu_{P_n}) = \text{mean}(\mu_{K_n}) = 0
\]

variance

\[
\text{var}(\mu_{P_n}) = \frac{2(n-1)}{n} \to 2, \quad \text{var}(\mu_{K_n}) = n - 1 \to \infty
\]

▶ In general, it is not reasonable to consider $\lim \mu_n$ when $\text{var}(\mu_n) \to \infty$.

We should take normalized limit $\lim \tilde{\mu}_n$. 
5.3. Formulation of Question in General

\( G_\nu = (V_\nu, E_\nu) \): growing graphs

\((\mathcal{A}(G_\nu), \langle \cdot \rangle_\nu)\): adjacency algebra with a state (algebraic probability space)

\( \mu_\nu \): spectral distribution of the adjacency matrix \( A_\nu \) of \( G_\nu \), i.e.,

\[
\langle A_\nu^m \rangle = \int_{-\infty}^{+\infty} x^m \mu_\nu(dx), \quad m = 0, 1, 2, \ldots .
\]

Note: \( \text{mean}(A_\nu) = \langle A_\nu \rangle \) and \( \text{var}(A_\nu) = \langle (A_\nu - \text{mean}(A_\nu))^2 \rangle \).

Main question in general

For the normalization \( \tilde{\mu}_\nu \) of \( \mu_\nu \) find the limit spectral distribution:

\[
\mu = \lim_\nu \tilde{\mu}_\nu .
\]

In other words,

\[
\lim_\nu \left\langle \left( \frac{A_\nu - \text{mean}(A_\nu)}{\sqrt{\text{var}(A_\nu)}} \right)^m \right\rangle_\nu = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 0, 1, 2, \ldots .
\]
5.4. Growing Distance-Regular Graphs (DRGs)

Definition
A graph $G = (V, E)$ is called distance regular if the intersection numbers:

$$p_{i,j}^k = |\{z \in V ; d(x, z) = i, d(y, z) = j\}|,$$

is constant for all pairs $x, y$ such that $d(x, y) = k$.

Examples: Hamming graphs, Johnson graphs, odd graphs, homogeneous trees, ...

We are interested in growing distance-regular graphs, e.g.,

$H(d, N)$ as $d \to \infty$ and $N \to \infty$

$J(v, d)$ as $v \to \infty$ and $d \to \infty$

$O_k$ as $k \to \infty$

$T_k$ as $k \to \infty$

\ldots
5.4. Growing Distance-Regular Graphs (DRGs)

Some general facts on a distance-regular graph $G$ (exercise)

1. Let $A = A^+ + A^- + A^\circ$ be the quantum decomposition (with respect to a fixed origin $o \in V$). Then

$$A^+ \Phi_n = \sqrt{\omega_n+1} \Phi_{n+1}, \quad A^- \Phi_n = \sqrt{\omega_n} \Phi_{n-1}, \quad A^\circ \Phi_n = \alpha_{n+1} \Phi_n,$$

where

$$\omega_n = p_{1,n-1}^{n-1} p_{1,n}^{n-1}, \quad \alpha_n = p_{1,n-1}^{n-1}.$$

2. In particular, $(\Gamma(G), \{\Phi_n\}, A^+ , A^\circ, A^-)$ is an IFS associated to $(\{\omega_n\}, \{\alpha_n\})$.

3. Mean value and variance:

$$\text{mean}(A) = \langle A \rangle = 0, \quad \text{var}(A) = \langle A^2 \rangle = \deg(o) = p_{11}^0.$$

4. If $G$ is a finite distance-regular graph, the tracial and vacuum states coincide:

$$\langle A^m \rangle_{\text{tr}} = \langle A^m \rangle_{\circ} = \langle e_o, A^m e_o \rangle, \quad m = 1, 2, \ldots.$$
5.5. Growing DRGs: An Example $H(d, N)$

$H(d, N) = K_N \times \cdots \times K_N$ ($d$ times): Hamming graph

\[ p^0_{1,1} = \deg H(d, N) = d(N - 1), \]
\[ p^n_{1,n-1} = n, \quad p^{n-1}_{1,n} = (d - n)(N - 1), \quad p^{n-1}_{1,n-1} = (n - 1)(N - 2). \]

**Theorem**

Let $\mu_{d,N}$ be the vacuum spectral distribution of $H(d, N)$ (in coincidence with the eigenvalue distribution). Then the Jacobi parameters of $\mu_{d,N}$ are given by

\[ \omega_n = p^n_{1,n-1} p^{n-1}_{1,n} = n(d - n + 1)(N - 1), \quad 1 \leq n \leq d, \]
\[ \alpha_n = p^{n-1}_{1,n-1} = (n - 1)(N - 2), \quad 1 \leq n \leq d + 1. \]

In fact, the vacuum spectral distribution of $A$ is the binomial distribution.

The IFS structure:

\[ A^+ \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1} = \sqrt{(n + 1)(d - n)(N - 1)} \Phi_{n+1}, \]
\[ A^- \Phi_n = \sqrt{\omega_n} \Phi_{n-1} = \sqrt{n(d - n + 1)(N - 1)} \Phi_{n-1}, \]
\[ A^0 \Phi_n = \alpha_{n+1} \Phi_n = n(N - 2) \Phi_n, \]
5.5. Growing DRGs: An Example $H(d, N)$

\[
A^+ \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1} = \sqrt{(n + 1)(d - n)(N - 1)} \Phi_{n+1},
\]
\[
A^- \Phi_n = \sqrt{\omega_n} \Phi_{n-1} = \sqrt{n(d - n + 1)(N - 1)} \Phi_{n-1},
\]
\[
A^\circ \Phi_n = \alpha_{n+1} \Phi_n = n(N - 2) \Phi_n,
\]

- What happens when $N \to \infty$ and $d \to \infty$?

- Normalization: $\text{mean}(A) = \langle A \rangle = 0$ and $\text{var}(A) = \langle A^2 \rangle = d(N - 1)$.

\[
\frac{A^+}{\sqrt{d(N - 1)}} \Phi_n = \sqrt{(n + 1) \left(1 - \frac{n}{d}\right)} \Phi_{n+1},
\]
\[
\frac{A^-}{\sqrt{d(N - 1)}} \Phi_n = \sqrt{n \left(1 - \frac{n - 1}{d}\right)} \Phi_{n-1},
\]
\[
\frac{A^\circ}{\sqrt{d(N - 1)}} \Phi_n = n \sqrt{\frac{N - 2}{d}} \sqrt{\frac{N - 2}{N - 1}} \Phi_n,
\]

- We thus find the proper scaling:

\[
N \to \infty, \quad d \to \infty, \quad \frac{N}{d} \to \tau \geq 0.
\]
5.5. Growing DRGs: An Example $H(d, N)$

- Taking the limit as $N \to \infty$, $d \to \infty$ and $\frac{N}{d} \to \tau \geq 0$, we have

$$A^+ \sqrt{\frac{d}{d(N-1)}} \Phi_n = \sqrt{(n+1)\left(1 - \frac{n}{d}\right)} \Phi_{n+1} \to \sqrt{n+1} \ \Phi_{n+1} \ 	ext{and}$$

$$A^- \sqrt{\frac{d}{d(N-1)}} \Phi_n = \sqrt{n\left(1 - \frac{n-1}{d}\right)} \Phi_{n-1} \to \sqrt{n} \ \Phi_{n-1} \ \text{and}$$

$$A^\circ \sqrt{\frac{d}{d(N-1)}} \Phi_n = n \sqrt{\frac{N-2}{d}} \sqrt{\frac{N-2}{N-1}} \Phi_n \to n \sqrt{\tau} \ \Phi_n \ .$$

- Recall the Boson Fock space $(\Gamma, \{\Psi_n\}, B^+, B^-)$ is defined by

$$B^+ \Psi_n = \sqrt{n+1} \Psi_{n+1}, \quad B^- \Psi_n = \sqrt{n} \Psi_{n-1} .$$

- Note also that

$$B^+ B^- \Psi_n = n \Psi_n .$$
5.5. Growing DRGs: An Example $H(d, N)$

**Theorem (Quantum central limit theorem (QCLT) for $H(d, N)$)**

Let $A = A^+ + A^- + A^\circ$ be the quantum decomposition of the adjacency matrix of $H(d, N)$. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be the Boson Fock space. Then we have

$$
\left( \frac{A^+}{\sqrt{d(N-1)}}, \frac{A^-}{\sqrt{d(N-1)}}, \frac{A^\circ}{\sqrt{d(N-1)}} \right) \xrightarrow{m} (B^+, B^-, \sqrt{\tau} B^+ B^-),
$$

as $N \to \infty$, $d \to \infty$ and $\frac{N}{d} \to \tau \geq 0$.

where $\xrightarrow{m}$ means the convergence of all mixed moments.

Deteiled proof is omitted (exercise).
5.5. Growing DRGs: An Example $H(d, N)$

Finding the asymptotic spectral distribution for $H(d, N)$

$$
\left( \frac{A^+}{\sqrt{d(N-1)}}, \frac{A^-}{\sqrt{d(N-1)}}, \frac{A^\circ}{\sqrt{d(N-1)}} \right) \xrightarrow{m} (B^+, B^-, \sqrt{\tau} B^+ B^-)
$$

implies that

$$
\langle e_o \left( \frac{A}{\sqrt{d(N-1)}} \right)^m e_o \rangle \xrightarrow{} \langle \Psi_0, (B^+ + B^- + \sqrt{\tau} B^+ B^-)^m \Psi_0 \rangle.
$$

On the other hand, by observing moments or generating functions, we see that

$$
\langle \Psi_0, (B^+ + B^- + \sqrt{\tau} B^+ B^-)^m \Psi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx),
$$

where

$$
\mu = \begin{cases} 
N(0, 1), & \tau = 0, \\
\text{affine transformed Po}(\tau^{-1}), & \tau > 0.
\end{cases}
$$

This $\mu$ is the asymptotic spectral (\equiv eigenvalue) distribution of $H(d, N)$. 
5.6. Growing DRGs: General Results

\{G_\nu\}: growing DRGs with adjacency matrices \(A_\nu\)

- Using mean\((A_\nu) = \langle A_\nu \rangle = 0\) and var\((A_\nu) = \langle A_\nu^2 \rangle = \deg(G_\nu) = p_{11}^0(\nu)\), the normalization of \(A_\nu\) is given by

\[
\frac{A_\nu - \text{mean}(A_\nu)}{\sqrt{\text{var}(A_\nu)}} = \frac{A_\nu^+}{\sqrt{\deg(G_\nu)}} + \frac{A_\nu^0}{\sqrt{\deg(G_\nu)}} + \frac{A_\nu^-}{\sqrt{\deg(G_\nu)}}.
\]

Theorem (Quantum CLT for growing DRGs)

Assume that for all \(n = 1, 2, \ldots\) the limits

\[
\omega_n = \lim_\nu p_{1,n-1}^n(\nu)p_{1,n-1}^{n-1}(\nu), \quad \alpha_n = \lim_\nu \frac{p_{1,n-1}^{n-1}(\nu)}{\sqrt{p_{1,1}^0(\nu)}},
\]

exist. Let \((\Gamma, \{\Phi_n\}, B^+, B^-, B^0)\) be the interacting Fock space associated with \((\{\omega_n\}, \{\alpha_n\})\). Then we have

\[
\left(\frac{A_\nu^+}{\sqrt{\deg(G_\nu)}}, \frac{A_\nu^-}{\sqrt{\deg(G_\nu)}}, \frac{A_\nu^0}{\sqrt{\deg(G_\nu)}}\right) \xrightarrow{m} (B^+, B^-, B^0).
\]
5.7. Growing Regular Graphs — Going Slightly Beyond DRGs

$Z^N$ as $N \to \infty$

1. $\Gamma(Z^N)$ is asymptotically invariant under $A^\epsilon$:

\[
A^+ \Phi_n = \sqrt{2N} \sqrt{n+1} \Phi_{n+1} + O(1),
\]
\[
A^- \Phi_n = \sqrt{2N} \sqrt{n} \Phi_{n-1} + O(N^{-1/2}).
\]

2. Normalized adjacency matrices:

\[
\frac{A^\epsilon_N}{\sqrt{\text{deg}(A_N)}} = \frac{A^\epsilon_N}{\sqrt{2N}} \to B^\epsilon
\]

3. The interacting Fock space in the limit:

\[
B^+ \Psi_n = \sqrt{n+1} \Psi_{n+1},
\]
\[
B^- \Phi_n = \sqrt{n} \Psi_{n-1}, \quad B^\circ = 0.
\]

This is Boson Fock space!

4. The asymptotic spectral distribution is the standard Gaussian distribution:

\[
\lim_{N \to \infty} \left \langle e_o, \left( \frac{A_N}{\sqrt{2N}} \right)^m e_o \right \rangle = \left \langle \Psi_0, (B^+ + B^-)^m \Psi_0 \right \rangle
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx.
\]
5.7. Growing Regular Graphs — Going Slightly Beyond DRGs

Statistics of $\omega_\epsilon(x)$

$$M(\omega_\epsilon|V_n) = \frac{1}{|V_n|} \sum_{x \in V_n} |\omega_\epsilon(x)|$$

$$\Sigma^2(\omega_\epsilon|V_n) = \frac{1}{|V_n|} \sum_{x \in V_n} \{ |\omega_\epsilon(x)| - M(\omega_\epsilon|V_n) \}^2$$

$$L(\omega_\epsilon|V_n) = \max\{|\omega_\epsilon(x)|; x \in V_n\}.$$ 

Conditions for growing regular graphs $G_\nu = (V^{(\nu)}, E^{(\nu)})$

(A1) $\lim_\nu \kappa(\nu) = \infty$, where $\kappa(\nu) = \text{deg}(G_\nu)$.

(A2) for each $n = 1, 2, \ldots$,

$$\exists \lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \quad \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty.$$ 

(A3) for each $n = 0, 1, 2, \ldots$,

$$\exists \lim_\nu \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_\nu \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_\nu \frac{L(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty.$$
5.7. Growing Regular Graphs — Going Slightly Beyond DRGs

Theorem (QCLT for growing regular graphs)

Let \( \{ G_\nu = (V^{(\nu)}, E^{(\nu)}) \} \) be a growing regular graph satisfying

(A1) \( \lim_\nu \kappa(\nu) = \infty \), where \( \kappa(\nu) = \deg(G_\nu) \).

(A2) for each \( n = 1, 2, \ldots \),

\[ \exists \lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \quad \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty. \]

(A3) for each \( n = 0, 1, 2, \ldots \),

\[ \exists \lim_\nu \frac{M(\omega_0|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_\nu \frac{\Sigma^2(\omega_0|V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_\nu \frac{L(\omega_0|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty. \]

Let \( (\Gamma, \{ \Psi_n \}, B^+, B^-, B^\circ) \) be the interacting Fock space associated with the Jacobi parameters \( (\{ \omega_n \}, \{ \alpha_n \}) \). Then

\[ \left( \frac{A^+_\nu}{\sqrt{\kappa(\nu)}}, \frac{A^-_\nu}{\sqrt{\kappa(\nu)}}, \frac{A^\circ_\nu}{\sqrt{\kappa(\nu)}} \right) \xrightarrow{m} (B^+, B^-, B^\circ) \]

In particular, the asymptotic spectral distribution of the normalized \( A_\nu \) in the vacuum state is a probability distribution determined by \( (\{ \omega_n \}, \{ \alpha_n \}) \).
Outline of Proof

(1) \[ \frac{A_{\epsilon}}{\sqrt{\kappa}} \Phi_n = \gamma_{n+\epsilon}^{\epsilon} \Phi_{n+\epsilon} + S_{n+\epsilon}^{\epsilon}, \quad \epsilon \in \{+,-,\circ\}, \quad n = 0, 1, 2, \ldots. \]

\[ \gamma_{n+} = M(\omega_- | V_n) \left( \frac{|V_n|}{\kappa |V_{n-1}|} \right)^{1/2}, \quad \gamma_{n-} = M(\omega_+ | V_n) \left( \frac{|V_n|}{\kappa |V_{n+1}|} \right)^{1/2}, \quad \gamma_{n} = \frac{M(\omega_0 | V_n)}{\sqrt{\kappa}}. \]

(2) \[ |V_n| = \left\{ \prod_{k=1}^{n} M(\omega_- | V_k) \right\}^{-1} \kappa^n + O(\kappa^{n-1}). \]

(3) \[ \lim_{\nu} \gamma_n^+ = \sqrt{\omega_n}, \quad \lim_{\nu} \gamma_n^- = \sqrt{\omega_{n+1}}, \quad \lim_{\nu} \gamma_n^\circ = \alpha_{n+1}. \]

(4) \[ \frac{A_{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A_{\epsilon_1}}{\sqrt{\kappa}} \Phi_n = \gamma_{n+\epsilon_1}^{\epsilon_1} \gamma_{n+\epsilon_1+\epsilon_2}^{\epsilon_2} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_m}^{\epsilon_m} \Phi_{n+\epsilon_1+\cdots+\epsilon_m} + \sum_{k=1}^{m} \gamma_{n+\epsilon_1}^{\epsilon_1} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_{k-1}}^{\epsilon_{k-1}} \cdot \left( \frac{A_{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A_{\epsilon_{k+1}}}{\sqrt{\kappa}} \right) S_{n+\epsilon_1+\cdots+\epsilon_{k}}^{\epsilon_{k}}. \]

(5) Estimate the error terms and show that

\[ \lim_{\nu} \left\langle \Phi_j^{(\nu)}, \frac{A_{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_{\epsilon_{k+1}}}{\sqrt{\kappa(\nu)}} S_{n+\epsilon_1+\cdots+\epsilon_{k}}^{\epsilon_{k}} \right\rangle = 0. \]
5.8. Deformed Vacuum States on $\mathcal{A}(G)$

Definition ($Q$-matrix and deformed vacuum functional)

The *$Q$-matrix* of a graph $G = (V, E)$ is defined by

$$Q = Q_q = [q^{d(x,y)}]_{x,y \in V}, \quad d(x, y) = \text{graph distance},$$

where $q$ is a parameter (in fact, we are interested only in the case of $-1 \leq q \leq 1$). The *deformed vacuum functional* is defined by

$$\langle a \rangle_q = \langle Q_q e_o, a e_o \rangle, \quad a \in \mathcal{A}(G).$$

1. For $q = 0$ we have $Q_0 = I$ so that $\langle \cdot \rangle_q$ coincides with the vacuum state.
2. $Q e_o$ does not necessarily belong to $\ell^2(V)$ but $\langle a \rangle_q$ is well-defined for $a \in \mathcal{A}(G)$.
3. $\mathcal{A}(G) \ni a \mapsto \langle a \rangle_q$ is a merely a *normalized linear function*.
4. Positivity of $\langle \cdot \rangle_q$ is an interesting question from several aspects.
5.8. Deformed Vacuum States on $\mathcal{A}(G)$

Let $G$ be a $\kappa$-regular graph and consider the deformed vacuum functional on $\mathcal{A}(G)$:

$$\langle a \rangle_q = \langle Q_q e_0, a e_0 \rangle, \quad a \in \mathcal{A}(G).$$

We have

$$\langle A \rangle_q = \kappa q,$$

$$\Sigma^2_q(A) = \langle (A - \langle A \rangle_q)^2 \rangle_q = \kappa (1 - q) \{1 + q + q M(\omega | V_1)\}$$

so that the quantum decomposition of the normalized adjacency matrix is given by

$$\frac{A - \langle A \rangle_q}{\Sigma_q(A)} = \frac{A^+}{\Sigma_q(A)} + \frac{A^-}{\Sigma_q(A)} + \frac{A^\circ - \langle A \rangle_q}{\Sigma_q(A)}$$

Let $\{G_\nu\}$ be growing regular graphs. We need to find a proper scaling balance between $\kappa(\nu)$ and $q(\nu)$.

The balance condition found from the actions of $A^e$ and explicit form of $Q_q e_0$:

$$\lim_{\nu} \kappa(\nu) = \infty, \quad \lim_{\nu} q(\nu) = 0, \quad \lim_{\nu} q(\nu) \sqrt{\kappa(\nu)} = \gamma \in \mathbb{R}.$$
5.8. Deformed Vacuum States on $A(G)$

(A1) $\lim_\nu \kappa(\nu) = \infty$, where $\kappa(\nu) = \text{deg}(G_\nu)$.

(A2) for each $n = 1, 2, \ldots$,

$\exists \lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \quad \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty$.

(A3) for each $n = 0, 1, 2, \ldots$,

$\exists \lim_\nu \frac{M(\omega_0|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_\nu \frac{\Sigma^2(\omega_0|V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_\nu \frac{L(\omega_0|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty$.

(A4) (scaling balance) $\lim_\nu q(\nu) = 0$ and $\lim_\nu q(\nu) \sqrt{\kappa(\nu)} = \gamma \in \mathbb{R}$ (constant).

Lemma

Under (A1)–(A4) we have

$$Qe_o = \sum_{n=0}^{\infty} q^n \sqrt{|V_n|} \Phi_n \rightarrow \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{\omega_n \cdots \omega_1}} \Psi_n = \Omega_\gamma$$

The above $\Omega_\gamma$ is reasonably called a coherent vector of the interacting Fock space since

$$B^- \Omega_\gamma = \gamma \Omega_\gamma.$$  

5.8. Deformed Vacuum States on $\mathcal{A}(G)$

**Theorem (Deformed QCLT for growing regular graphs)**

Let $\{G_\nu = (V^{(\nu)}, E^{(\nu)})\}$ be a growing regular graph satisfying conditions (A1)–(A3) and $A_\nu$ its adjacency matrix. Let $(\Gamma, \{\Psi_n\}, B^+, B^-, B^\circ)$ be the IFS associated to $\{\omega_n\}, \{\alpha_n\}$. Under (A4) we have

$$\lim_{\kappa \to \infty, q \to 0} \frac{\tilde{A}^{\epsilon_m}}{\Sigma_q(A)} \cdots \frac{\tilde{A}^{\epsilon_1}}{\Sigma_q(A)} e_0 = \langle \Omega_\gamma, \tilde{B}^{\epsilon_m} \cdots \tilde{B}^{\epsilon_1} \Psi_0 \rangle,$$

where

$$\tilde{A}^\pm = A_\nu^\pm, \quad \tilde{A}^\circ = A_\nu^\circ - \langle A_\nu \rangle_q, \quad \tilde{B}^\pm = \frac{B^\pm}{\sqrt{1 + \gamma \alpha_2}}, \quad \tilde{B}^\circ = \frac{B^\circ - \gamma}{\sqrt{1 + \gamma \alpha_2}}.$$

In particular,

$$\lim_{\kappa \to \infty, q \to 0} \frac{A_\nu - \langle A \rangle_q}{\Sigma_q(A_\nu)}^m_q = \langle \Omega_\gamma, \left( \frac{B^+ + B^- + B^\circ - \gamma}{\sqrt{1 + \gamma \alpha_2}} \right)^m \Psi_0 \rangle.$$

▶ **Challenging Exercise:** Examine the above argument for $T_\kappa$ as $\kappa \to \infty$ and find the limit distribution (free Poisson distribution $\Rightarrow$ Marchenko–Pastur distribution).
Some concrete examples: Asymptotic spectral distributions

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<th>graphs</th>
<th>IFS</th>
<th>vacuum state</th>
<th>deformed vacuum state</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamming graphs $H(d, N)$</td>
<td>$\omega_n = n$ (Boson)</td>
<td>Gaussian ($N/d \to 0$)</td>
<td>Gaussian or Poisson</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Poisson ($N/d \to \lambda^{-1} &gt; 0$)</td>
<td></td>
</tr>
<tr>
<td>Johnson graphs $J(v, d)$</td>
<td>$\omega_n = n^2$</td>
<td>exponential ($2d/v \to 1$)</td>
<td>‘Poissonization’ of exponential distribution</td>
</tr>
<tr>
<td></td>
<td></td>
<td>geometric ($2d/v \to p \in (0, 1)$)</td>
<td></td>
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<tr>
<td>odd graphs $O_k$</td>
<td>$\omega_{2n-1} = n$</td>
<td>two-sided Rayleigh</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td>$\omega_{2n} = n$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>homogeneous trees $T_\kappa$</td>
<td>$\omega_n = 1$ (free)</td>
<td>Wigner semicircle</td>
<td>free Poisson</td>
</tr>
<tr>
<td>integer lattices $\mathbb{Z}^N$</td>
<td>$\omega_n = n$ (Boson)</td>
<td>Gaussian</td>
<td>Gaussian</td>
</tr>
<tr>
<td>symmetric groups $S_n$ (Coxeter)</td>
<td>$\omega_n = n$ (Boson)</td>
<td>Gaussian</td>
<td>Gaussian</td>
</tr>
<tr>
<td>Coxeter groups (Fendler)</td>
<td>$\omega_n = 1$ (free)</td>
<td>Wigner semicircle</td>
<td>free Poisson</td>
</tr>
<tr>
<td>Spidernets $S(a, b, c)$</td>
<td>$\omega_1 = 1$</td>
<td>free Meixner law</td>
<td>(free Meixner law)</td>
</tr>
<tr>
<td></td>
<td>$\omega_2 = \cdots = q$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
6. Concepts of Independence and Graph Products
6.1. (Classical) Independence and Central Limit Theorem

$X, Y, \ldots$: random variables on a classical probability space $(\Omega, \mathcal{F}, P)$

**Definition**

Two random variables $X$ and $Y$ are called *independent* if

$$P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b), \quad a, b \in \mathbb{R}.$$  

**Theorem (multiplicativity of mean values)**

*If two random variables $X, Y$ are independent, then*

$$E[XY] = E[X]E[Y].$$

*Moreover,*

$$E[X^mY^n] = E[X^m]E[Y^n]$$

*whenever the mean values exist.*
6.1. (Classical) Independence and Central Limit Theorem

\( X_1, X_2, \ldots \): sequence of random variables such that

(i) independent

(ii) identically distributed

(iii) normalized, i.e., \( \mathbb{E}[X_n] = 0, \mathbb{V}[X_n] = \mathbb{E}[X_n^2] = 1 \)

- Law of Large Numbers (LLN) says that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} X_n = 0 \quad \text{almost surely.}
\]

- Central Limit Theorem (CLT) describes the fluctuation of

\[
\lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n
\]
6.1. (Classical) Independence and Central Limit Theorem

Theorem (Central limit theorem (CLT))

Let $X_1, X_2, \ldots$ be a sequence of random variables such that (i) independent, (ii) identically distributed, and (iii) normalized. Then

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n$$

obeys the standard normal distribution $N(0, 1)$ in the limit.

$$\lim_{N \to \infty} P \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \leq a \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx,$$

or equivalently, for any $f \in C_b(\mathbb{R})$,

$$\lim_{N \to \infty} \mathbb{E} \left[ f \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right) \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-x^2/2} dx.$$
6.1. (Classical) Independence and Central Limit Theorem

**Theorem (Algebraic Version of CLT)**

Let $X_1, X_2, \ldots$ be a sequence of random variables such that (i) independent, (ii) identically distributed, and (iii) normalized. If $X_n$ has finite moments of all orders, we have

$$\lim_{N \to \infty} E \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right)^m \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx.$$

In other words,

$$\lim_{N \to \infty} E \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right)^{2m} \right] = \frac{(2m)!}{2^m m!},$$

$$\lim_{N \to \infty} E \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right)^{2m-1} \right] = 0.$$
6.1. (Classical) Independence and Central Limit Theorem

Combinatorial Proof

\[
E\left( \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right)^m \right) = \frac{1}{N^{m/2}} \sum_{n_1, \ldots, n_m=1}^{N} E[X_{n_1} X_{n_2} \cdots X_{n_m}] 
\]

► We pick up the essential terms \( E[X_{n_1} X_{n_2} \cdots X_{n_m}] \) that contributes to the limit.

1) \( E[X_{n_1} X_{n_2} \cdots X_{n_m}] = E[X_i] E[\cdots \cdots] = 0. \) \( \exists X_i \) appears only once

2) Hence we only need to count the terms

\[
E\left[ X_{n_1} X_{n_2} \cdots \cdots X_{n_m} \right] \quad \# \text{ of distinct } X_i \text{'s} \leq \left\lfloor \frac{m}{2} \right\rfloor
\]
6.1. (Classical) Independence and Central Limit Theorem

\[ E \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right)^m \right] = \frac{1}{N^{m/2}} \sum_{n_1, \ldots, n_m=1}^{N} E[X_{n_1} X_{n_2} \cdots X_{n_m}] \]

Hence we only need to count the terms

\[ E \left[ \underbrace{X_{n_1} X_{n_2} \cdots \cdots X_{n_m}}_{\# \ of \ distinct \ X_i's \leq \left[ \frac{m}{2} \right]} \right] \]

Let \( s \) be the number of distinct \( X_i \)'s. The number of such terms is

\[ \binom{N}{s} \times \#\{\text{arrangements of } X_1, \ldots, X_s\} \sim N^s C(s). \]

Thus the terms of \( s < m/2 \) have no contribution in the limit.

Namely, only the terms of \( s = m/2 \) have contribution in the limit.
6.1. (Classical) Independence and Central Limit Theorem

\[
E \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right)^m \right] = \frac{1}{N^{m/2}} \sum_{n_1, \ldots, n_m=1}^{N} E[X_{n_1} X_{n_2} \cdots X_{n_m}]
\]

Namely, only the terms of \( s = m/2 \) have contribution in the limit.

If \( m \) is odd,

\[
\lim_{N \to \infty} E \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right)^m \right] = 0.
\]

Suppose that \( m = 2s \) is even.

\[
E[X_{n_1} X_{n_2} \cdots X_{n_m}] = E[X_{i_1}^2 X_{i_2}^2 \cdots X_{i_s}^2] = E[X_{i_1}^2]E[X_{i_2}^2] \cdots E[X_{i_s}^2] = 1.
\]

Each distinct \( X_i \)'s appears twice

Consequently,

\[
\lim_{N \to \infty} E \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right)^{2s} \right] = \lim_{N \to \infty} \frac{1}{N^s} \left( \begin{array}{c} N \\ s \end{array} \right) \frac{(2s)!}{2^s} = \frac{(2s)!}{2^s s!}.
\]
6.2. Independence in Quantum Probability and Quantum CLT

- Algebraic version of CLT is proved by
  - using factorization rule of mixed moments $E[X_{n_1}X_{n_2} \cdots X_{n_m}]$, 
  - picking up the essential terms that contribute to the limit.

Factorization rule

- For classical random variables $X$ and $Y$, obviously we have
  \[
  \]

- But for $a = a^*$, $b = b^*$ in $(A, \varphi)$ we wonder
  \[
  \varphi(baa) \neq \varphi(aba) \neq \varphi(aab) = ??? \quad \ldots
  \]
  There are many possibilities arising from non-commutativity.

Our viewpoint

- Independence is formulated as a “good” factorization rule.
- There are four basic concepts of independence in quantum probability.
6.2. Independence in Quantum Probability and Quantum CLT

Suppose we are given a concept of *independence* in \((\mathcal{A}, \varphi)\).

Then we may consider a sequence \(\{a_n\}\) of random variables in \((\mathcal{A}, \varphi)\) such that

1. **real**, i.e., \(a_n = a_n^*\),
2. **independent**,
3. **identically distributed**,
4. **normalized**, i.e., \(\varphi(a_n) = 0\) and \(\varphi(a_n^2) = 1\).

Then we ask for the probability distribution \(\mu\) such that

\[
\lim_{N \to \infty} \varphi\left[\left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n\right)^m\right] = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots
\]

We call \(\mu\) the *central limit distribution*. 
### 6.2. Independence in Quantum Probability and Quantum CLT

#### Four Concepts of Independence and Quantum CLTs

- Factorization rules are shown only for three mixed moments of low orders.

<table>
<thead>
<tr>
<th></th>
<th>commutative</th>
<th>free</th>
<th>Boolean</th>
<th>monotone</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varphi(aba))</td>
<td>(\varphi(a^2)\varphi(b))</td>
<td>(\varphi(a^2)\varphi(b))</td>
<td>(\varphi(a)^2\varphi(b))</td>
<td>(\varphi(a^2)\varphi(b))</td>
</tr>
<tr>
<td>(\varphi(bab))</td>
<td>(\varphi(a)\varphi(b^2))</td>
<td>(\varphi(a)\varphi(b^2))</td>
<td>(\varphi(a)\varphi(b)^2)</td>
<td>(\varphi(a)\varphi(b)^2)</td>
</tr>
<tr>
<td>(\varphi(abab))</td>
<td>(\varphi(a^2)\varphi(b^2))</td>
<td>(\varphi(a)^2\varphi(b^2)) + (\varphi(a^2)\varphi(b)^2)</td>
<td>(\varphi(a)^2\varphi(b)^2)</td>
<td>(\varphi(a^2)\varphi(b)^2)</td>
</tr>
<tr>
<td>CLM</td>
<td>Gaussian</td>
<td>Wigner</td>
<td>Bernoulli</td>
<td>arcsine</td>
</tr>
</tbody>
</table>
6.2. Independence in Quantum Probability and Quantum CLT

► One more: \( \varphi(a_2a_1a_4a_3a_4a_3a_6a_6a_4a_4a_3a_5) = \varphi(214343664435) \)

[commutative independence]

\[
\varphi(214343664435) = \varphi(1)\varphi(2)\varphi(3^3)\varphi(4^4)\varphi(5)\varphi(6^2)
\]

[monotone independence]

\[
\varphi(214343664435) = \varphi(2)\varphi(4)\varphi(4)\varphi(66)\varphi(133443)\varphi(5) \\
= \varphi(2)\varphi(4)\varphi(4)\varphi(66)\varphi(44)\varphi(1333)\varphi(5) \\
= \varphi(2)\varphi(4)\varphi(4)\varphi(66)\varphi(44)\varphi(333)\varphi(1)
\]

[Boolean independence]

\[
\varphi(214343664435) = \varphi(2)\varphi(1)\varphi(4)\varphi(3)\varphi(4)\varphi(3)\varphi(66)\varphi(44)\varphi(3)\varphi(5)
\]
6.2. Independence in Quantum Probability and Quantum CLT

Central limit distributions

\[ \varphi \left[ \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k \right)^m \right] \to \int_{-\infty}^{+\infty} x^m \mu(dx). \]

Theorem (QCLT)

[commutative CLT] If \( a_1, a_2, \ldots \) are commutative independent, we have

\[ \mu(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \quad \text{(normal distribution)} \]

[monotone CLT] If \( a_1, a_2, \ldots \) are monotone independent, we have

\[ \mu(dx) = \frac{dx}{\pi \sqrt{2 - x^2}} \quad \text{(normalized arcsine law)} \]

[Boolean CLT] If \( a_1, a_2, \ldots \) are Boolean independent, we have

\[ \mu = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1} \quad \text{(normalized Bernoulli distribution)} \]
### 6.2. Independence in Quantum Probability and Quantum CLT

**Outline of proof**

\[
\varphi\left[\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k\right)^m\right] = \frac{1}{n^{m/2}} \sum_{k_1, \ldots, k_m=1}^{n} \varphi[a_{k_1} a_{k_2} \cdots a_{k_m}]
\]

- We pick up the essential terms \( \varphi[a_{k_1} a_{k_2} \cdots a_{k_m}] \) that contributes to the limit.

1. \( \varphi(a_{k_1} a_{k_2} \cdots a_{k_m}) = 0 \) if there is a singleton.

2. \( \varphi(a_{k_1} a_{k_2} \cdots a_{k_m}) \) contributes to the limit only if the number \( s \) of distinct \( a_i \)'s is \( s = \lfloor m/2 \rfloor \).

3. According to the independence evaluate \( \varphi(a_{k_1} a_{k_2} \cdots a_{k_m}) \), where distinct \( a_i \)'s appear exact twice.
6.2. Independence in Quantum Probability and Quantum CLT

Outline of proof

Finally we get

$$\lim_{n \to \infty} \varphi \left[ \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k \right)^{2m-1} \right] = 0$$

for three cases and

$$\lim_{n \to \infty} \varphi \left[ \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k \right)^{2m} \right] = \begin{cases} 
\frac{(2m)!}{2^m m!}, & \text{commutative independence,} \\
\frac{(2m)!}{2^m m! m!}, & \text{monotone independence,} \\
1, & \text{Boolean independence.}
\end{cases}$$

Cf. free CLT

$$\lim_{n \to \infty} \varphi \left[ \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k \right)^{2m} \right] = \frac{1}{m + 1} \binom{2m}{m} = \int_{-2}^{2} x^m \frac{1}{2\pi} \sqrt{4 - x^2} \, dx.$$
6.3. Graph Products

A binary operation of graphs

\[ (G_1, G_2) \mapsto \Phi(G_1, G_2) = G_1 \# G_2 \]

\[ (A_1, A_2) \mapsto \Phi(A_1, A_2) = A[G_1 \# G_2] \]

\[ (\mu_1, \mu_2) \mapsto \Phi(\mu_1, \mu_2) = \mu_1 \# \mu_2 \text{ (convolution)} \]
6.3. Graph Products — Cartesian Product

**Definition**

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* or *direct product* of $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is a graph on $V = V_1 \times V_2$ with adjacency relation:

$$(x, y) \sim (x', y') \iff \begin{cases} x = x' & \text{or} \quad x \sim x' \\ y \sim y' & \quad y = y'. \end{cases}$$

**Example** ($C_4 \times C_3$)

![Diagram showing the Cartesian product of $C_4$ and $C_3$]
### 6.3. Graph Products — Comb Product

**Definition**

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We fix a vertex $o_2 \in V_2$. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

(i) $x = x'$ and $y \sim y'$; (ii) $x \sim x'$ and $y = y' = o_2$.

Then $V_1 \times V_2$ becomes a graph, denoted by $G_1 \triangleright_o G_2$, and is called the **comb product** or the **hierarchical product**.

**Example ($C_4 \triangleright_o C_3$ with $o_2 = 1'$)**

![Diagram of Comb Product]

The diagram illustrates the comb product $C_4 \triangleright C_3$ with $o_2 = 1'$.
6.3. Graph Products — Star Product

Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with distinguished vertices $o_1 \in V_1$ and $o_2 \in V_2$. Define a subset of $V_1 \times V_2$ by

$$V_1 \star V_2 = \{(x, o_2) ; x \in V_1\} \cup \{(o_1, y) ; y \in V_2\}$$

The induced subgraph of $G_1 \times G_2$ spanned by $V_1 \star V_2$ is called the star product of $G_1$ and $G_2$ (with contact vertices $o_1$ and $o_2$), and is denoted by $G_1 \star G_2 = G_1_{o_1} \star_{o_2} G_2$.

Example ($C_4 \star C_3$)

\[
\begin{align*}
C_4 & \quad \quad 3' \\
1' & \quad \quad 2'
\end{align*}
\]

\[
\begin{align*}
(1,1') & \quad (1,3') \\
(1,2') & \quad (4,1') \\
(2,1') & \quad (3,1')
\end{align*}
\]
6.3. Graph Products — Adjacency Matrices

$G_1 = (V_1, E_1), G_2 = (V_2, E_2)$: two graphs

$G = G_1 \# G_2$: a graph product and assume that $V[G] = V_1 \times V_2$

$A_i = A[G_i]$: adjacency matrix of $G_i$ acting on $\ell^2(V_i), (i = 1, 2)$

$\implies A = A[G_1 \# G_2]$ acts on

$$\ell^2(V) = \ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2).$$

**Theorem**

- **[Cartesian product]**

  $$A[G_1 \times G_2] = A_1 \otimes I_2 + I_1 \otimes A_2.$$

- **[comb product]**

  $$A[G_1 \triangleright G_2] = A_1 \otimes P_2 + I_1 \otimes A_2.$$

- **[star product]**

  $$A[G_1 \star G_2] = A_1 \otimes P_2 + P_1 \otimes A_2.$$

Here, $P_i$ is the rank one projection corresponding to $o_i$. 
6.4. Quantum CLT for Graph Products

Let $\phi_i$ be the vacuum state at $o_i$ and consider the product state $\varphi = \varphi_1 \otimes \varphi_2$.

$\implies A = A[G_1 \# G_2]$ is a random variable in $\langle \mathcal{A}(G_1 \# G_2), \varphi \rangle$.

**Theorem**

Let $A_i = A[G_i]$ be the adjacency matrix of $G_i$.

1. **[Cartesian product]**

   $A[G_1 \times G_2] = A_1 \otimes I_2 + I_1 \otimes A_2$

   is a sum of **commutative independent** random variables.

2. **[comb product]**

   $A[G_1 \triangleright G_2] = A_1 \otimes P_2 + I_1 \otimes A_2$

   is a sum of **monotone independent** random variables.

3. **[star product]**

   $A[G_1 \star G_2] = A_1 \otimes P_2 + P_1 \otimes A_2$

   is a sum of **Boolean independent** random variables.
6.4. Quantum CLT for Graph Products

**Associativity of graph operations**

1. **[Cartesian product]**

\[(G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3)\]

2. **[Comb product]**

\[(G_1 \triangleright G_2) \triangleright G_3 \cong G_1 \triangleright (G_2 \triangleright G_3)\]

   To be precise,

\[(G_1 \triangleright_{o_2} G_2) \triangleright_{o_3} G_3 \cong G_1 \triangleright_{(o_2,o_3)} (G_2 \triangleright_{o_3} G_3)\]

3. **[Star product]**

\[(G_1 \star G_2) \star G_3 \cong G_1 \star (G_2 \star G_3)\]

▶ Thus, we have naturally \(n\)-fold powers:

\[G^{\#n} = G \# G \# \cdots \# G \quad (n \text{ times})\]

\[A[G^{\#n}] = B_1 + B_2 + \cdots + B_n\]
6.4. Quantum CLT for Graph Products

Theorem (CLT for Cartesian product graphs)

For the $n$-fold Cartesian power $G^{(n)} = G \times \cdots \times G$ ($n$-times),
\[
\lim_{n \to \infty} \left\langle \left( \frac{A^{(n)}}{\sqrt{n} \sqrt{\deg(o)}} \right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx.
\]

Theorem (CLT for comb product graphs)

For the $n$-fold monotone power $G^{(n)} = G \rhd_o G \rhd_o \cdots \rhd_o G$ ($n$-times),
\[
\lim_{n \to \infty} \left\langle \left( \frac{A^{(n)}}{\sqrt{n} \sqrt{\deg(o)}} \right)^m \right\rangle = \int_{-\sqrt{2}}^{+\sqrt{2}} x^m \frac{dx}{\pi \sqrt{2 - x^2}}, \quad m = 1, 2, \ldots.
\]

Theorem (CLT for star product graphs)

For the $n$-fold star power $G^{(n)} = G \ast G \ast \cdots \ast G$ ($n$-times) we have
\[
\lim_{n \to \infty} \left\langle \left( \frac{A^{(n)}}{\sqrt{n} \sqrt{\deg(o)}} \right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m \frac{1}{2} (\delta_{-1} + \delta_{+1}) (dx), \quad m = 1, 2, \ldots.
\]
## More Graph Products

<table>
<thead>
<tr>
<th>products</th>
<th>$G_1 # G_2$</th>
<th>$A[G_1 # G_2]$</th>
<th>spectral distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian</td>
<td>$G_1 \times_C G_2$</td>
<td>$A_1 \otimes I_2 + I_1 \otimes A_2$</td>
<td>$\mu_1 \ast \mu_2$</td>
</tr>
<tr>
<td>monotone</td>
<td>$G_1 \triangleright G_2$</td>
<td>$A_1 \otimes P_2 + I_2 \otimes A_2$</td>
<td>$\mu_1 \triangleright \mu_2$</td>
</tr>
<tr>
<td>star</td>
<td>$G_1 \star G_2$</td>
<td>$A_1 \otimes P_2 + P_1 \otimes A_2$</td>
<td>$\mu_1 \uplus \mu_2$</td>
</tr>
<tr>
<td>lexicographic</td>
<td>$G_1 \triangleright_L G_2$</td>
<td>$A_1 \otimes J_2 + P_1 \otimes A_2$</td>
<td>$D(\mu_1) \triangleright \mu_2$</td>
</tr>
<tr>
<td>Kronecker</td>
<td>$G_1 \times_K G_2$</td>
<td>$A_1 \otimes A_2$</td>
<td>$\mu_1 \ast_M \mu_2$</td>
</tr>
<tr>
<td>strong</td>
<td>$G_1 \times_S G_2$</td>
<td>$A_1 \otimes I_2 + I_1 \otimes A_2 + A_1 \otimes A_2$</td>
<td>$S^{-1}(S\mu_1 \ast_M S\mu_2)$</td>
</tr>
<tr>
<td>free</td>
<td>$G_1 \star G_2$</td>
<td>$A_1 \ast A_2$</td>
<td>$\mu_1 \uplus \mu_2$</td>
</tr>
</tbody>
</table>

1. Every product except the free product is a graph on $V_1 \times V_2$.
2. There is a classification of graph products realized on $V_1 \times V_2$, see e.g., R. Hammack et al.: “Handbook of Product Graphs,” CRC Press, 2011.
Exercise 12  Let $G_n$ be the graph obtained by joining $n$ triangles ($K_3 \cong C_3$ at the origin $o$, also called the $n$-fold star product of $K_3$. (The following figure shows $G_6$.) Calculate explicitly the spectral distribution of $G_n$ at $o$ and study its asymptotic behavior as $n \to \infty$. 

![Graph Image]

7.1. Counting Walks and Spectral Distributions

$G = (V, E)$: a (finite or infinite) graph

$o \in V$: a fixed origin

$W_m(o; G) = |\{o \to o : m\text{-step walk}\}|$

**Theorem**

Let $A$ be the adjacency matrix of $G$ and $\mu$ the vacuum spectral distribution at $o \in V$. Then we have

$$W_m(o; G) = \langle e_o, A^m e_o \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 0, 1, 2, \ldots.$$

טרים wir are interested in the correspondence

$$G \rightarrow \mu$$

from the point of view of counting walks.
7.1. Counting Walks and Spectral Distributions

Basic result (1) $\mathbb{Z}$

$$W_{2m}(0; \mathbb{Z}) = \binom{2m}{m} = \int_{-2}^{2} x^{2m} \alpha(dx), \quad \alpha(x) = \frac{1}{\pi \sqrt{4 - x^2}}.$$  

Basic result (2) $\mathbb{Z}_+ = \{0, 1, 2, \ldots \}$

$$W_{2m}(0; \mathbb{Z}_+) = \frac{1}{m + 1} \binom{2m}{m} = \int_{-2}^{2} x^{2m} w(dx), \quad w = \frac{1}{2\pi} \sqrt{4 - x^2}.$$  

Catalan number
7.2. Cartesian Product: \( W((o_1, o_2); G_1 \times C G_2) \)

The adjacency matrix of \( G_1 \times C G_2 \) is
\[
A = A_1 \otimes I + I \otimes A_2,
\]
where two matrices in RHD are commutative.

We then have
\[
\langle e_{(o_1, o_2)}, A^m e_{(o_1, o_2)} \rangle = \langle e_{o_1} \otimes e_{o_2}, (A_1 \otimes I + I \otimes A_2)^m e_{o_1} \otimes e_{o_2} \rangle = \sum_{k=0}^{m} {m \choose k} \langle e_{o_1} \otimes e_{o_2}, A_1^k \otimes A_2^{m-k} e_{o_1} \otimes e_{o_2} \rangle = \sum_{k=0}^{m} {m \choose k} \langle e_{o_1}, A_1^k e_{o_1} \rangle \langle e_{o_2} \otimes A_2^{m-k} e_{o_2} \rangle
\]

Consequently,
\[
W((o_1, o_2); G_1 \times C G_2) = \sum_{k=0}^{m} {m \choose k} W_k(o_1; G_1) W_{m-k}(o_2; G_2)
\]
7.2. Cartesian Product: \( W((o_1, o_2); G_1 \times_C G_2) \)

\( \mu_i \): Spectral distribution of \( G_i \) at \( o_i \)

\( \mu \): Spectral distribution of \( G = G_1 \times_C G_2 \) at \( (o_1, o_2) \)

\( W_m(o_i; G_i) = \int_{-\infty}^{+\infty} x^m \mu_i(dx), \quad W_m((o_1, o_2); G_1 \times_C G_2) = \int_{-\infty}^{+\infty} x^m \mu(dx). \)

Then the identity

\[
W((o_1, o_2); G_1 \times_C G_2) = \sum_{k=0}^{m} \binom{m}{k} W_k(o_1; G_1) W_{m-k}(o_2; G_2)
\]

implies that

\[
\int_{-\infty}^{+\infty} x^m \mu(dx) = \sum_{k=0}^{m} \binom{m}{k} \int_{-\infty}^{+\infty} x^k \mu_1(dx) \int_{-\infty}^{+\infty} x^{m-k} \mu_2(dx)
\]

\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x_1 + x_2)^m \mu_1(dx_1) \mu_2(dx_2).
\]

Thus, \( \mu = \mu_1 \ast \mu_2 \) (classical) convolution.
### 7.3. Graph Products and Convolution of Distributions

<table>
<thead>
<tr>
<th>products</th>
<th>$G_1 # G_2$</th>
<th>$A[G_1 # G_2]$</th>
<th>spectral distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cartesian comb</td>
<td>$G_1 \times_C G_2$</td>
<td>$A_1 \otimes I_2 + I_1 \otimes A_2$</td>
<td>$\mu_1 \ast \mu_2$</td>
</tr>
<tr>
<td>star</td>
<td>$G_1 \triangleright G_2$</td>
<td>$A_1 \otimes P_2 + I_2 \otimes A_2$</td>
<td>$\mu_1 \triangleright \mu_2$</td>
</tr>
<tr>
<td>lexicographic</td>
<td>$G_1 \triangleright_L G_2$</td>
<td>$A_1 \otimes J_2 + P_1 \otimes A_2$</td>
<td>$\mu_1 \uplus \mu_2$</td>
</tr>
<tr>
<td>Kronecker</td>
<td>$G_1 \times_K G_2$</td>
<td>$A_1 \otimes A_2$</td>
<td>$D(\mu_1) \triangleright \mu_2$</td>
</tr>
<tr>
<td>strong</td>
<td>$G_1 \times_S G_2$</td>
<td>$A_1 \otimes I_2 + I_1 \otimes A_2$</td>
<td>$\mu_1 \ast_M \mu_2$</td>
</tr>
<tr>
<td>free</td>
<td>$G_1 \ast G_2$</td>
<td>$A_1 \ast A_2$</td>
<td>$S^{-1}(S\mu_1 \ast_M S\mu_2)$</td>
</tr>
</tbody>
</table>

1. Every product except the free product is a graph on $V_1 \times V_2$.
2. There is a classification of graph products realized on $V_1 \times V_2$, see e.g., R. Hammack et al.: “Handbook of Product Graphs,” CRC Press, 2011.
7.3. Graph Products and Convolution of Distributions

- Monotone convolution $\mu = \mu_1 \triangleright \mu_2$ is characterized by

$$H_\mu(z) = H_{\mu_1}(H_{\mu_2}(z)),$$

where

$$H_\mu(z) = \frac{1}{G_\mu(z)}, \quad G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x}.$$

- Boolean convolution $\mu = \mu_1 \oplus \mu_2$ is characterized by

$$\frac{1}{G_\mu(z)} = \frac{1}{G_{\mu_1}(z)} + \frac{1}{G_{\mu_2}(z)} - z.$$
7.4. Kronecker Product

Definition (Kronecker product)

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The Kronecker product $G_1 \times_K G_2$ is a graph on $V = V_1 \times V_2$ with the adjacency relation:

$$(x, y) \sim_K (x', y') \iff x \sim x', \ y \sim y'.$$

In other words, the adjacency matrix $A = A[G_1 \times_K G_2]$ is given by

$$A = A_1 \otimes A_2.$$  

1. $G_1 \times_K G_2 \cong G_2 \times_K G_1$.

2. $(G_1 \times_K G_2) \times_K G_3 \cong G_1 \times_K (G_2 \times_K G_3)$.

3. (trivial case) For any graph $G = (V, E)$ the Kronecker product $K_1 \times_K G$ is a graph on $V$ with no edges (i.e., an empty graph on $V$).
7.4. Kronecker Product

**Lemma (exercise)**

If $|V_1| \geq 2$ and $|V_2| \geq 2$, then $G_1 \times_K G_2$ has at most two connected components.

**Lemma (exercise)**

$G_1 \times_K G_2$ is a subgraph of the distance-2 graph of $G_1 \times_C G_2$. (But not necessarily induced subgraph.)
7.5. Counting Walks in Kronecker Product

\( G_i = (V_i, E_i) \): a connected graph with fixed origin \( o_i \in V_i \)

\( G = G_1 \times_K G_2 \): Kronecker product with origin \((o_1, o_2)\)

\( G^o = (G_1 \times_K G_2)^o \): the connected component containing \((o_1, o_2)\)

\[
W_m((o_1, o_2); G) = W_m((o_1, o_2); G^o) \\
= \langle e_{(o_1, o_2)}, A^m e_{(o_1, o_2)} \rangle \\
= \langle e_{o_1} \otimes e_{o_2}, (A_1 \otimes A_2)^m e_{o_1} \otimes e_{o_2} \rangle \\
= \langle e_{o_1}, A_1^m e_{o_1} \rangle \langle e_{o_2}, A_2^m e_{o_2} \rangle \\
= W_m(o_1; G_1) W_m(o_2; G_2)
\]
7.5. Counting Walks in Kronecker Product

\( G_i = (V_i, E_i) \): a connected graph with fixed origin \( o_i \in V_i \)

\( G = G_1 \times_K G_2 \): Kronecker product with origin \((o_1, o_2)\)

\( G^o = (G_1 \times_K G_2)^o \): the connected component containing \((o_1, o_2)\)

Thus,

\[
W_m((o_1, o_2); G) = W_m(o_1; G_1)W_m(o_1; G_2).
\]

\( \mu_i \): spectral distribution of the adjacency matrix \( A_i \) at \( o_i \)

\( \mu \): spectral distribution of the adjacency matrix \( A = A[G] \) at \((o_1, o_2)\)

\[
\int_{-\infty}^{+\infty} x^m \mu(dx) = \int_{-\infty}^{+\infty} x_1^m \mu_1(dx_1) \int_{-\infty}^{+\infty} x_2^m \mu_2(dx_2)
\]

\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x_1 x_2)^m \mu_1(dx_1) \mu_2(dx_2)
\]

This \( \mu \) is called the Mellin convolution and denoted by \( \mu = \mu_1 *_M \mu_2 \).
7.5. Counting Walks in Kronecker Product

**Theorem**

For $i = 1, 2$ let $G_i = (V_i, E_i)$ be a graph with a distinguished vertex $o_i$. Let $\mu_i$ be the spectral distribution of the adjacency matrix $A_i = A[G_i]$ at $o_i$. Then the spectral distribution of $G = G_1 \times_K G_2$ at $(o_1, o_2)$ is given by the Mellin convolution:

$$\mu(G_1 \times_K G_2) = \mu_1 *_M \mu_2.$$ 

- $\delta_a *_M \delta_b = \delta_{ab}$ for $a, b \in \mathbb{R}$.
  
  [cf. $\delta_a * \delta_b = \delta_{a+b}$.]

- If $\mu_i(dx) = f_i(x)dx$ and $f_i(-x) = f_i(x)$, then $\mu_1 *_M \mu_2$ admits a symmetric density function $2f_1 * f_2(x)$, where

$$f_1 * f_2(x) = \int_0^\infty f_1(y) f_2\left(\frac{x}{y}\right) \frac{dy}{y} = \int_0^\infty f_1\left(\frac{x}{y}\right) f_2(y) \frac{dy}{y}, \quad x > 0.$$ 

In fact, this is the standard convolution of the multiplicative group $\mathbb{R}_{>0}$. 

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Exercises

Exercise 13  Observe that $(K_2 \times K_2)\circ \cong K_2$ and examine the identity:

$$
\left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\right) * M \left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\right) = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1.
$$

Exercise 14  Using $K_3 \times K_2 \cong C_6$, derive the spectral distribution of $C_6$ at a fixed origin (which in fact coincides with the eigenvalue distribution):

$$
\frac{1}{6}\delta_{-2} + \frac{1}{3}\delta_{-1} + \frac{1}{3}\delta_1 + \frac{1}{6}\delta_2.
$$

Exercise 15  Using $K_4 \times K_2 \cong K_2 \times C K_2 \times C K_2 = H(3, 2)$, derive the spectral distribution of $H(3, 2)$ at a fixed origin (which in fact coincides with the eigenvalue distribution):

$$
\frac{1}{8}\delta_{-3} + \frac{3}{8}\delta_{-1} + \frac{3}{8}\delta_1 + \frac{1}{8}\delta_3.
$$

Also examine the identity:

$$
\left(\frac{3}{4}\delta_{-1} + \frac{1}{4}\delta_3\right) * M \left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\right) = \left(\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1\right)^3.
$$
7.6. Restricted Lattices

- \( \mathbb{Z} \times_C \mathbb{Z} \) (2d integer lattice): a graph on \( \mathbb{Z}^2 \) with adjacency relation:
  \[
  (x, y) \sim (x', y') \iff \begin{cases} 
  x' = x \pm 1, & \text{or} \\
  y' = y,
  \end{cases}
  \]

- \( \mathbb{Z} \times_K \mathbb{Z} \): a graph on \( \mathbb{Z}^2 = \{(u, v) \); \( u, v \in \mathbb{Z} \} \) with adjacency relation:
  \[
  (u, v) \sim_K (u', v') \iff u' = u \pm 1 \quad \text{and} \quad v' = v \pm 1.
  \]

\( \mathbb{Z} \times_K \mathbb{Z} \) has two connected components, each of which is isomorphic to \( \mathbb{Z} \times_C \mathbb{Z} \).

Let \( (\mathbb{Z} \times_K \mathbb{Z})^O \) denote the connected component of \( \mathbb{Z} \times_K \mathbb{Z} \) containing \( O = (0, 0) \). Then
  \[
  (\mathbb{Z} \times_K \mathbb{Z})^O \cong \mathbb{Z} \times_C \mathbb{Z}.
  \]
7.6. Restricted Lattices

\( \mathbb{Z} \times_K \mathbb{Z} \) has two connected components, each of which is isomorphic to \( \mathbb{Z} \times_C \mathbb{Z} \).

Let \( (\mathbb{Z} \times_K \mathbb{Z})^O \) denote the connected component of \( \mathbb{Z} \times_K \mathbb{Z} \) containing \( O = (0, 0) \). Then

\[
(\mathbb{Z} \times_K \mathbb{Z})^O \cong \mathbb{Z} \times_C \mathbb{Z}.
\]

Since the spectral distribution of \( \mathbb{Z} \) at \( 0 \) is the arcsine law \( \alpha \), we have

**Theorem**

The spectral distribution of 2d lattice \( \mathbb{Z}^2 \) at \( (0, 0) \) is given by

\[
\alpha *_M \alpha = \alpha * \alpha
\]
7.6. Restricted Lattices

Let $L\{x \geq y\}$ denote the induced subgraph of $\mathbb{Z} \times_{C} \mathbb{Z}$ spanned by the vertices

$$\{(x, y) \in \mathbb{Z}^2; x \geq y\}.$$

**Theorem**

We have $L\{x \geq y\} \cong (\mathbb{Z}_+ \times K \mathbb{Z})^O$ and its spectral distribution at $(0, 0)$ is given by

$$\mathbf{w} \ast_M \alpha.$$
7.6. Restricted Lattices

Let $L\{x \geq y \geq -x\}$ denote the induced subgraph of $\mathbb{Z} \times \mathbb{C} \mathbb{Z}$ spanned by the vertices

$$\{(x, y) \in \mathbb{Z}^2; x \geq y \geq -x\}.$$ 

**Theorem**

We have $L\{x \geq y \geq -x\} \cong (\mathbb{Z}_+ \times \mathbb{Z})^O$ and its spectral distribution at $(0, 0)$ is given by

$$\mathbf{w} * \mathbf{M} * \mathbf{w}$$
## 7.6. Restricted Lattices

<table>
<thead>
<tr>
<th>Domain $D$</th>
<th>$W_{2m}(L[D], O)$</th>
<th>spectral distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}$</td>
<td>$\binom{2m}{m}$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\mathbb{Z}_+$</td>
<td>$C_m = \frac{1}{m+1}\binom{2m}{m}$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>$\mathbb{Z}^2$</td>
<td>$\binom{2m}{m}^2$</td>
<td>$\alpha \ast \alpha = \alpha \ast_M \alpha$</td>
</tr>
<tr>
<td>${x \geq y}$</td>
<td>$C_m(\binom{2m}{m})$</td>
<td>$\omega \ast_M \alpha$</td>
</tr>
<tr>
<td>${x \geq y \geq -x}$</td>
<td>$C_m^2$</td>
<td>$\omega \ast \omega$</td>
</tr>
<tr>
<td>${x \geq 0, y \geq 0}$</td>
<td>(A)</td>
<td>$\pi_n \ast_M \alpha$</td>
</tr>
<tr>
<td>${x \geq y \geq x - (n - 1)}$</td>
<td>(B)</td>
<td>$\pi_k \ast_M \pi_l$</td>
</tr>
<tr>
<td>$\left{\begin{array}{l}0 \leq x + y \leq k - 1, \0 \leq x - y \leq l - 1\end{array}\right.$</td>
<td>(C)</td>
<td></td>
</tr>
</tbody>
</table>

(A) \[= \sum_{k=0}^{m} \binom{2m}{2k} C_k C_{m-k},\]

(B) \[= W_{2m}(P_n, 0)\binom{2m}{m},\]

(C) \[= W_{2m}(P_k, 0) W_{2m}(P_l, 0).\]
7.6. Restricted Lattices — Density Functions

**Elliptic integrals** For $k^2 < 1$, the elliptic integrals are defined by

\[
K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}},
\]

\[
E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^1 \frac{1 - k^2 x^2}{\sqrt{1 - x^2}} dx.
\]

The density function of $w *_M \alpha$ is given by

\[
\frac{1}{\pi^2} \{K(\xi(x)) - E(\xi(x))\}, \quad \xi(x) = \sqrt{1 - \frac{x^2}{16}}, \quad -4 \leq x \leq 4.
\]

The density function of $\alpha *_M \alpha = \alpha * \alpha$ is given by

\[
\frac{1}{2\pi^2} K(\xi(x)), \quad -4 \leq x \leq 4.
\]

The density function of $w *_M w$ is given by

\[
\frac{2}{\pi^2} \left\{ \left(1 + \frac{x^2}{16}\right) K(\xi(x)) - 2E(\xi(x)) \right\}, \quad -4 \leq x \leq 4.
\]
7.6. Restricted Lattices — Density Functions

\[ w * M \alpha \]

\[ \alpha * M \alpha \]

\[ w * M w \]
An Example in 3-Dimension: $\mathbb{Z} \times K \mathbb{Z} \times K \mathbb{Z}$

$\mathbb{Z} \times K \mathbb{Z} \times K \mathbb{Z}$ has 4 connected components, which are mutually isomorphic. The connected component containing $O(0, 0, 0)$ looks like an octahedra honeycomb, built up by gluing octahedra or body-centered cubes.

We have

$$W_{2m}(\mathbb{Z} \times K \mathbb{Z} \times K \mathbb{Z}, (0, 0, 0)) = \binom{2m}{m}^3, \quad m = 0, 1, 2, \ldots,$$

and the spectral distribution is given by $\mu = \alpha *_M \alpha *_M \alpha$. 

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8. Bivariate Extension: An Example

Motivation

(I) Quantum CLT: $A_\nu \xrightarrow{m} B$

⇒ The limit spectral distribution is a probability distribution on $\mathbb{R}^1$

⇒ Multi-variate extension: $(A_\nu^{(1)}, \ldots, A_\nu^{(p)}) \xrightarrow{m} (Z_1, \ldots, Z_p)$?

See e.g., T. Espinasse and P. Rochet (2019), arXiv:1904.10720

— An extension of Boolean CLT

(II) Method of quantum decomposition $A = A^+ + A^\circ + A^-$

⇒ Orthogonal polynomials in one variable:

\[ xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_nP_{n-1}(x) \]

⇒ Multi-variate extension?

potentially very interesting in connection to multi-variate orthogonal polynomials
8.1. Hamming Graphs $H(n, v)$

- $n \geq 1$, $v \geq 1$: natural numbers
- Alphabets $K = \{1, 2, \ldots, v\}$
- Words of length $n$:

$$V = \{x = (\xi_1, \xi_2, \ldots, \xi_n) \mid \xi_i \in K\} = K^n$$

- Hamming distance between two words $x$ and $y$:

$$\partial(x, y) = |\{1 \leq i \leq n \mid \xi_i \neq \eta_i\}|.$$  

- A graph is defined with vertex set $V$ and adjacency relation

$$x \sim y \iff \partial(x, y) = 1$$

$\Rightarrow$ This is the Hamming graph $H(n, v)$. 
8.1. Hamming Graphs

- **Product structure**

\[ H(n, v) = K_v \times \cdots \times K_v \quad (n\text{-fold Cartesian power}) \]

where \( K_v \) is the complete graph on \( v \) vertices.

- **The adjacency matrix of** \( H(n, v) \) **is given by**

\[
A_{n,v} = \sum_{i=1}^{n} I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I,
\]

where \( A = A[K_v] \) is the adjacency matrix of \( K_v \).

- **The eigenvalue distribution** \( \mu_{n,v} \) **is specified by**

\[
\frac{1}{vn} \text{Tr}(A_{n,v}^m) = \int_{-\infty}^{+\infty} x^m \mu_{n,v}(dx), \quad m = 0, 1, 2, \ldots.
\]

**Question [CLT for Hamming graphs]**

\[ \mu_{n,v} \rightarrow ?? \quad \text{as} \quad n \rightarrow \infty \quad \text{and} \quad v \rightarrow \infty \]
8.1. Hamming Graphs

Review of Hora’s argument (1998). This is before quantum decomposition

The adjacency matrix of $K_v$ is given by $A = J - I$ ($J$: all-one matrix)

Then $C(K_v) = \mathbb{C}^v = U_{v-1} \oplus U_{-1}$ and

$$A \upharpoonright U_{v-1} = v - 1, \quad \dim U_{v-1} = 1, \quad A \upharpoonright U_{-1} = -1, \quad \dim U_{-1} = v - 1.$$ 

$A_{n,v} = \sum I \otimes \cdots \otimes A \otimes \cdots \otimes I$ acts on

$$(\mathbb{C}^v)^{\otimes n} = (U_{v-1} \oplus U_{-1}) \otimes \cdots \otimes (U_{v-1} \oplus U_{-1})$$

The eigenvalues of $A_{n,v}$ are $(v - 1)(n - j) + (-1)^j = -n + (n - j)v$

with multiplicity $\binom{n}{j} 1^{n-j} (v - 1)^{n-j}$, where $0 \leq j \leq n$.

Hence

$$\mu_{n,v} = \frac{1}{v^n} \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} (v - 1)^{n-j} \delta_{-n+(n-j)v}$$

$$= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{v}\right)^k \left(1 - \frac{1}{v}\right)^{n-k} \delta_{-n+vk}$$
8.1. Hamming Graphs

\[ \mu_{n,v} \text{ is essentially the binomial distribution:} \]

\[ \mu_{n,v} = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{v} \right)^k \left( 1 - \frac{1}{v} \right)^{n-k} \delta_{n+v+k} \]

By classical theory we know

\[ B(n, p) \approx N(np, np(1 - p)), \quad B(n, p) \approx Po(np) \]

mean(\(\mu_{n,v}\)) = 0, var(\(\mu_{n,v}\)) = \(n(v - 1)\) ⇒ normalized distribution \(\tilde{\mu}_{n,v}\)

Under the proper scaling \(n \to \infty, v \to \infty\) and \(\frac{v}{n} \to \tau \geq 0\),

\[ \tilde{\mu}_{n,v} \to \begin{cases} N(0, 1), & \tau = 0, \\ \text{affine transform of } Po(\tau^{-1}), & \tau > 0 \end{cases} \]

▶ Actual proof is based on characteristic functions (Laplace transform).
8.2. Strongly Regular Graphs

In general, $\bar{G}$ denotes the complementary graph of $G = (V, E)$, i.e., a graph on $V$ with edge set $\bar{E} = \{\{x, y\}; x, y \in V, x \neq y, \{x, y\} \notin E\}$.

Or equivalently, the adjacency matrix of $\bar{G}$ is defined by

$$\bar{A} = J - I - A.$$  \hspace{1cm} (J: all-one matrix)

**Lemma**

For a finite graph $G$ with adjacency matrix $A$ we have

$G$ is a regular graph $\iff A\bar{A} = \bar{A}A \iff AJ = JA.$

**Definition**

For a finite regular graph $G$ the commutative $*$-algebra generated by $I, A, \bar{A}$, denoted by $\mathcal{A}(G, \bar{G})$, is called the “extended adjacency algebra.”
8.2. Strongly Regular Graphs

Definition

\( G = (V, E) \) is a strongly regular graph with parameter \( (v, k, \lambda, \mu) \) if

1. \( |V| = v \);
2. \( G \) is \( k \)-regular;
3. every two adjacent \( x, y \in V \) has \( \lambda \) common adjacent vertices;
4. every two non-adjacent \( x, y \in V \) has \( \mu \) common adjacent vertices;
5. (avoiding trivial cases) \( G \) is neither complete nor empty, that is, \( 0 < k < v - 1 \).

Note: A strongly regular graph is a distance-regular graph with diameter 2.
8.2. Strongly Regular Graphs

Lemma

If $G$ is a strongly regular graph with parameter $(v, k, \lambda, \mu)$, so is $\overline{G}$ with parameter $(v, \overline{k} = v - k - 1, \overline{\lambda} = v - 2k + \mu - 2, \overline{\mu} = v - 2k + \lambda)$.

Lemma

Let $G$ be a finite regular graph with degree $0 < \kappa < v - 1$. Then the following conditions are equivalent:

1. $G$ is a strongly regular graph;
2. $\mathcal{A}(G, \overline{G})$ is the three-dimensional linear space spanned by $I, A, \overline{A}$.

For the proof we need only to note that

$$A^2 = kI + \lambda A + \mu \overline{A} = kI + \lambda A + \mu (J - I - A).$$
8.2. Strongly Regular Graphs

Lemma

Let $G$ be a strongly regular graph with $(v, k, \lambda, \mu)$. Then the spectrum of $G$ are given by

$$s < r < k$$

with multiplicities $g, f, 1$,

where

$$s, r = \frac{(\lambda - \mu) \pm \sqrt{\lambda^2 + 4(k - \mu)}}{2},$$

and

$$f = \frac{(v - 1)s + k}{s - r}, \quad g = \frac{(v - 1)r + k}{r - s}.$$

The spectrum of $\bar{G}$ are given by

$$\bar{s} = -r - 1 < \bar{r} = -s - 1 \leq \bar{k}$$

with multiplicities $f, g, 1$.

There are many relations among these constants. For example,

$$1 + k + \bar{k} = 1 + f + g = v, \quad k^2 + fr^2 + gs^2 = kv.$$
Let $G$ be a strongly regular graph and $\bar{G}$ the complement.

Consider the pair $(G^n, \bar{G}^n)$, where

$$G^n = G \times \cdots \times G \quad (\text{n-fold Cartesian power}), \quad \bar{G}^n = \bar{G} \times \cdots \times \bar{G} \quad \text{(similar)}.$$

Adjacency matrices:

$$A_{n,G} = \sum_{k=1}^{n} \left( I \boxtimes \cdots \boxtimes I \boxtimes A \boxtimes I \boxtimes \cdots \boxtimes I \right), \quad \bar{A}_{n,G} = (\text{similar}).$$

Let $\nu_{n,G}(dxdy)$ be the joint spectral distribution of $(A_{n,G}, \bar{A}_{n,G})$ specified by

$$\frac{1}{\nu^n} \text{Tr}(A_{n,G}^s \bar{A}_{n,G}^t) = \int_{\mathbb{R}^2} x^s y^t \nu_{n,G}(dxdy), \quad s, t = 0, 1, 2, \ldots.$$ 

Question (Asymptotic spectral distribution)

$$\nu_{n,G} \rightarrow ?? \quad \text{as} \quad n \rightarrow \infty \quad \text{and} \quad |G| \rightarrow \infty$$
8.3. Cartesian Product of Strongly Regular Graphs

How we generalized the case of Hamming graphs?

► Outline of our procedure:

1. Consider a strongly regular graph $G$ and its complement $\bar{G}$.
2. Consider a pair of Cartesian powers $(G^n, \bar{G}^n)$ and their adjacency matrices $(A_{n,G}, \bar{A}_{n,G})$.
3. The joint spectral distribution of $(A_{n,G}, \bar{A}_{n,G})$ is a probability distribution on $\mathbb{R}^2$ specified by

$$\frac{1}{v^n} \text{Tr}(A_{n,G}^s \bar{A}_{n,G}^t) = \int_{\mathbb{R}^2} x^s y^t \nu_{n,G}(dxdy), \quad s, t = 0, 1, 2, \ldots.$$ 

► Case of Hamming graphs:

Take $G = K_v$. Then $\bar{G}$ is an empty graph,

$$G^n = K_v \times \cdots \times K_v = H(n, v) \quad \text{(Hamming graph)},$$

$$\begin{align*}
(A_{n,G}, \bar{A}_{n,G}) &= (A_{n,v}, 0).
\end{align*}$$

Thus, the spectral distribution is reduced to one-dimension.
8.4. Joint spectral distribution of \((G^n, \bar{G}^n)\)

**Theorem**

The joint spectral distribution of \((G^n, \bar{G}^n)\) is given by

\[
\nu_{n,G} = \sum_{0 \leq j + h \leq n} \pi(j, h) \delta(\theta_{j,h}, \bar{\theta}_{j,h}), \\
\pi(j, h) = \binom{n}{j, h} \left( \frac{f}{v} \right)^j \left( \frac{g}{v} \right)^h \left( \frac{1}{v} \right)^{n-j-h},
\]

\[
\theta_{j,h} = (n - j - h)k + jr + hs, \\
\bar{\theta}_{j,h} = (n - j - h)\bar{k} + j\bar{s} + h\bar{r},
\]

\[
f = \frac{(v-1)s+k}{s-r}, \\
g = \frac{(v-1)r+k}{r-s}.
\]

**Proof:** According to \(\text{ev}(A_{n,G}) = \{s, r, k\}\) and \(\text{ev}(\bar{A}_{n,G}) = \{\bar{s}, \bar{s}, \bar{k}\}\) we have

\[
C(G) = \mathbb{C}^v = U_r \oplus U_s \oplus U_k, \\
\dim U_r = f, \\
\dim U_s = g, \\
\dim U_k = 1.
\]

Then look at

\[
A_{n,G} = \sum I \otimes \cdots \otimes A \otimes \cdots \otimes I,
\]

\[
C(G^n) = (U_r \oplus U_s \oplus U_k) \otimes \cdots \otimes (U_r \oplus U_s \oplus U_k).
\]
8.4. Joint spectral distribution of \((G^n, \bar{G}^n)\)

\[
\nu_{n,G} = \sum_{0 \leq j + h \leq n} \pi(j, h) \delta(\theta_{j,h}, \bar{\theta}_{j,h})
\]

\[
\pi(j, h) = \binom{n}{j, h} \left(\frac{f}{v}\right)^j \left(\frac{g}{v}\right)^h \left(\frac{1}{v}\right)^{n-j-h}
\]

\[
\nu_n, G = X_0 \leq j + h \leq n \pi(j, h) \delta(\theta_{j,h}, \bar{\theta}_{j,h})
\]

\[
\pi(j, h) = \binom{n}{j, h} \left(\frac{f}{v}\right)^j \left(\frac{g}{v}\right)^h \left(\frac{1}{v}\right)^{n-j-h}
\]

\[
ev(G) = \{s, r, k\}
\]

\[
ev(\bar{G}) = \{\bar{s}, \bar{r}, \bar{k}\}
\]
8.5. Asymptotic Joint Spectral Distributions

$n \to \infty, v \to \infty$ and some balance conditions

- **Hamming graphs:** $H(n, v) = K_v \times \cdots \times K_v$ ($n$-fold Cartesian power)
  
  \[
  \frac{v}{n} \to \tau \quad \text{and automatically} \quad \frac{-1}{n} \to 0, \quad \frac{v - 1}{n} \to \tau.
  \]
  
  these are conditions for eigenvalues!

- **Growing pair of strongly regular graphs:** $(G^n, \bar{G}^n)$

  Recall: $\text{ev}(G) = \{s, r, k\}$, $\text{ev}(\bar{G}) = \{\bar{r}, \bar{s}, \bar{k}\}$

  \[
  1 + k + \bar{k} = v, \quad \bar{s} = -r - 1, \quad \bar{r} = -s - 1.
  \]

  The proper scaling is given by

  \[
  \frac{k}{n} \to \kappa, \quad \frac{\bar{k}}{n} \to \bar{\kappa}, \quad \frac{r}{n} \to \rho, \quad \frac{s}{n} \to \sigma, \quad \frac{v}{n} \to \kappa + \bar{\kappa} \equiv \omega.
  \]

- **Note:** $\rho = 0$ or $\sigma = 0$ follows.
8.5. Asymptotic Joint Spectral Distributions

Theorem (Morales-Obata-Tanaka (2019+))

Let \( \nu \) be the limit of the joint spectral distribution of \( \left( \frac{A_{n,G}}{\sqrt{n}k}, \frac{\bar{A}_{n,G}}{\sqrt{n}k} \right) \). Then,

1. If \( \kappa > 0, \bar{\kappa} = -\sigma > 0, \rho = 0 \), then \( \nu \) is an affine transformation of the bivariate Poisson distribution:

\[
\nu \left( \left( \frac{\kappa j - \bar{\kappa} h}{\sqrt{\kappa}}, \frac{\bar{\kappa} j + \kappa h - 1}{\sqrt{\bar{\kappa}}} \right) \right) = e^{-1/\bar{\kappa}} \left( \frac{1}{\omega} \right)^j \left( \frac{\kappa}{\omega \bar{\kappa}} \right)^h \frac{1}{j! h!}
\]

2. If \( \kappa = \rho > 0, \bar{\kappa} > 0, \sigma = 0 \), then similar as above.

3. If \( \kappa > 0 \) or \( \bar{\kappa} > 0 \), and if \( \rho = \sigma = 0 \), then \( \nu \) is an affine transformation of the product of Gaussian and Poisson distributions:

\[
\int_{\mathbb{R}^2} f(x) \nu(dx) = \sqrt{\frac{\omega}{2\pi}} e^{-1/\omega} \sum_{h=0}^{\infty} \left( \frac{1}{\omega} \right)^h \frac{1}{h!} \int_{-\infty}^{+\infty} f(x_{h,t}) e^{-\omega t^2/2} dt
\]

\[
x_{h,t} = \left( \sqrt{\kappa} h + \sqrt{\bar{\kappa}} t - \frac{\sqrt{\kappa}}{\omega}, \sqrt{\bar{\kappa}} h - \sqrt{\kappa} t - \frac{\sqrt{\bar{\kappa}}}{\omega} \right)
\]

4. If \( \kappa = \bar{\kappa} = \rho = \sigma = 0 \), \( \nu \) is the bivariate Gaussian distribution.
8.5. Asymptotic Joint Spectral Distributions

Bivariate Poisson distribution

\[ \nu \left( \left( \frac{\kappa j - \bar{\kappa}h}{\sqrt{\kappa}}, \frac{\bar{\kappa}j + \bar{\kappa}h - 1}{\sqrt{\kappa}} \right) \right) = e^{-1/\bar{\kappa}} \left( \frac{1}{\omega} \right)^j \left( \frac{\kappa}{\omega \bar{\kappa}} \right)^h \frac{1}{j! h!} \]
8.5. Asymptotic Joint Spectral Distributions

**Gauss × Poisson distribution**

\[
\int_{\mathbb{R}^2} f(x) \nu(dx) = \sqrt{\frac{\omega}{2\pi}} e^{-1/\omega} \sum_{h=0}^{\infty} \left( \frac{1}{\omega} \right)^h \frac{1}{h!} \int_{-\infty}^{+\infty} f(x_{h,t}) e^{-\omega t^2 / 2} dt
\]

\[
x_{h,t} = \left( \sqrt{\kappa} h + \sqrt{\bar{\kappa}} t - \frac{\sqrt{\kappa}}{\omega} , \sqrt{\bar{\kappa}} h - \sqrt{\bar{\kappa}} t - \frac{\sqrt{\bar{\kappa}}}{\omega} \right)
\]
8.6. Bivariate Orthogonal Polynomials

Extended Adjacency Algebra $\mathcal{A}(G^n, \bar{G}^n)$

For $0 \leq \alpha + \beta \leq n$ we put

$$A_{\alpha,\beta} = \sum I \otimes \cdots \otimes A \otimes \cdots \otimes \bar{A} \otimes \cdots \otimes I,$$

$A$ appears $\alpha$ times and $\bar{A}$ appears $\beta$ times

In particular, the adjacency matrices of $(G^n, \bar{G}^n)$ are

$$A[G^n] = A_{n,G} = A_{1,0}, \quad A[\bar{G}^n] = \bar{A}_{n,G} = \bar{A}_{0,1}.$$

$\mathcal{A}(G^n, \bar{G}^n)$: unital $*$-algebra generated by $A_{n,G}$ and $\bar{A}_{n,G}$.

Lemma

$\mathcal{A}(G^n, \bar{G}^n)$ is a linear span of $\{A_{\alpha,\beta} ; 0 \leq \alpha + \beta \leq n\}$.

Lemma (Orthogonal relation)

$$\frac{1}{vn} \text{Tr}(A_{\alpha,\beta} A_{\alpha',\beta'}) = k_{\alpha,\alpha'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}, \quad k_{\alpha,\beta} = \binom{n}{\alpha, \beta} k^\alpha \bar{k}^\beta.$$
8.6. Bivariate Orthogonal Polynomials

Lemma (Mizukawa–Tanaka (PAMS 2004))

The eigenvalues of $A_{\alpha, \beta}$ are given in the form:

$$k_{\alpha, \beta} P_{\alpha, \beta}(j, h) \quad \text{with multiplicity} \quad \binom{n}{j, h} f^j g^h,$$

Bivariate Krawtchouk Polynomials

$$P_{\alpha, \beta}(j, h) = \sum_{0 \leq \nu_1 + \cdots + \nu_4 \leq n} \frac{(-\alpha)_{\nu_1 + \nu_3} (-\beta)_{\nu_2 + \nu_4} (-j)_{\nu_1 + \nu_2} (-h)_{\nu_3 + \nu_4}}{(-n)_{\nu_1 + \nu_2 + \nu_3 + \nu_4}} t_1^{\nu_1} t_2^{\nu_2} t_3^{\nu_3} t_4^{\nu_4} \frac{\nu_1! \nu_2! \nu_3! \nu_4!}{\nu_1! \nu_2! \nu_3! \nu_4!},$$

where

$$t_1 = 1 - \frac{r}{k}, \quad t_2 = 1 - \frac{s}{k}, \quad t_2 = 1 - \frac{s}{k}, \quad t_4 = 1 - \frac{r}{k}.$$

- This is a particular case of Aomoto-Gelfand hypergeometric function of $(3, 6)$-type.
- Pochhammer symbol: $(a)_n = a(a + 1)(a + 2) \cdots (a + n - 1)$
8.6. Bivariate Orthogonal Polynomials

Then the orthogonal relation becomes

\[ \sum_{0 \leq j + h \leq n} \sqrt{k_{\alpha,\beta}} P_{\alpha,\beta}(j, h) \sqrt{k_{\alpha',\beta'}} P_{\alpha',\beta'}(j, h) \pi(j, h) = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}. \]

Using integral form and applying variable change:

\[ \nu_{n,G} = \sum_{0 \leq j + h \leq n} \binom{n}{j, h} \pi(j, h) \delta(\theta_{j,h}, \bar{\theta}_{j,h}), \]

\[ \theta_{j,h} = (n - j - h)k + jr + hs, \quad \bar{\theta}_{j,h} = (n - j - h)\bar{k} + j\bar{s} + h\bar{r}, \]

\[ x = \frac{\theta_{j,h}}{\sqrt{nk}}, \quad y = \frac{\bar{\theta}_{j,h}}{\sqrt{nk}}, \]

we obtain polynomials \( \{ \tilde{P}_{\alpha,\beta}(x, y) \} \) such that

\[ \int_{\mathbb{R}^2} \tilde{P}_{\alpha,\beta}(x, y) \tilde{P}_{\alpha',\beta'}(x, y) \nu_{G,n}(dx dy) = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}. \]
8.7. Bivariate Orthogonal Polynomials in the Limit

▶ We consider the Gauss × Poisson case

Let

\[ R_{\alpha, \beta}(x, y) = \lim \tilde{P}_{\alpha, \beta}(x, y) \]

under the scaling

\[ \frac{k}{n} \to \kappa > 0 \lor \frac{\bar{k}}{n} \to \bar{\kappa} > 0, \quad \frac{r}{n} \to \rho = 0, \quad \frac{s}{n} \to \sigma = 0, \]

Then we have

\[ \int_{\mathbb{R}^2} R_{\alpha, \beta}(x, y) R_{\alpha', \beta'}(x, y) \nu(dx \, dy) = \delta_{\alpha, \alpha'} \delta_{\beta, \beta'} \]

Theorem (Morales-Obata-Tanaka (2019+))

\[ \{R_{\alpha, \beta}(x, y)\} \text{ are the orthogonal polynomials with respect to the Gauss × Poisson distribution } \nu. \]
### 8.7. Bivariate Orthogonal Polynomials in the Limit

#### Explicit form

1. We start with the generating function:

   \[ \sum_{0 \leq \alpha + \beta \leq n} k_{\alpha,\beta} P_{\alpha,\beta}(j, h) \xi_1^\alpha \xi_2^\beta \]

   \[ = (1 + k_1 \xi_1 + \bar{k}_2 \xi_2)^{n-j-h} (1 + r \xi_1 + s \xi_2)^j (1 + s \xi_1 + \bar{r} \xi_2)^h \]

2. Changing variables and taking the limit, we have

   \[ \sum_{\alpha, \beta = 0}^\infty \frac{R_{\alpha,\beta}(x, y)}{\sqrt{\alpha!\beta!}} \xi_1^\alpha \xi_2^\beta \]

   \[ = (1 + \sqrt{\kappa} \xi_1 + \sqrt{\kappa} \xi_2)^{(\sqrt{\kappa} x + \sqrt{\kappa} y + 1)/\omega} \]

   \[ \times \exp \left\{ - \frac{\sqrt{\kappa} \xi_1 + \sqrt{\kappa} \xi_2}{\omega} - \frac{(\sqrt{\kappa} \xi_1 - \sqrt{\kappa} \xi_2)^2}{2\omega} \right. \]

   \[ \left. + \frac{(\sqrt{\kappa} x - \sqrt{\kappa} y)(\sqrt{\kappa} \xi_1 - \sqrt{\kappa} \xi_2)}{\omega} \right\} \]
8.7. Bivariate Orthogonal Polynomials in the Limit

Five-term recurrence relation

1. We start with

\[ AA_{\alpha,\beta} = (\alpha + 1)A_{\alpha+1,\beta} + (\alpha + 1)(\bar{k} - \bar{\mu})A_{\alpha+1,\beta-1} \]
\[ + (\alpha \lambda + \beta (k - \mu))A_{\alpha,\beta} + (\beta + 1)\mu A_{\alpha-1,\beta+1} \]
\[ + (n - \alpha - \beta + 1)k A_{\alpha-1,\beta}, \]

\[ \bar{A}A_{\alpha,\beta} = (\beta + 1)A_{\alpha,\beta+1} + (\alpha + 1)\bar{\mu} A_{\alpha+1,\beta-1} \]
\[ + (\alpha(\bar{k} - \bar{\mu}) + \beta \bar{\lambda})A_{\alpha,\beta} + (\beta + 1)(k - \lambda)A_{\alpha-1,\beta+1} \]
\[ + (n - \alpha - \beta + 1)\bar{k} A_{\alpha,\beta-1}. \]

2. Use the correspondence:

\[ \frac{A_{\alpha,\beta}}{\sqrt{k_{\alpha,\beta}}} \leftrightarrow \sqrt{k_{\alpha,\beta}} P_{\alpha,\beta}(j, h) \]

we obtain the five-term recurrence relation for \( \{ P_{\alpha,\beta}(j, h) \} \).

3. Changing variables and taking the limit, we have
8.7. Bivariate Orthogonal Polynomials in the Limit

Theorem (Five-term recurrence relation)

\[ x R_{\alpha,\beta} = \sqrt{\alpha + 1} R_{\alpha+1,\beta} + \sqrt{(\alpha + 1)\beta} \frac{\kappa \sqrt{\kappa}}{\omega} R_{\alpha+1,\beta-1} \]
\[ + (\alpha \kappa + \beta \bar{\kappa}) \frac{\sqrt{\kappa}}{\omega} R_{\alpha,\beta} + \sqrt{\alpha(\beta + 1)} \frac{\kappa \sqrt{\kappa}}{\omega} R_{\alpha-1,\beta+1} + \sqrt{\alpha} R_{\alpha-1,\beta}, \]

\[ y R_{\alpha,\beta} = \sqrt{\beta + 1} R_{\alpha,\beta+1} + \sqrt{(\alpha + 1)\beta} \frac{\kappa \sqrt{\kappa}}{\omega} R_{\alpha+1,\beta-1} \]
\[ + (\alpha \kappa + \beta \bar{\kappa}) \frac{\sqrt{\kappa}}{\omega} R_{\alpha,\beta} + \sqrt{\alpha(\beta + 1)} \frac{\bar{\kappa} \sqrt{\kappa}}{\omega} R_{\alpha-1,\beta+1} + \sqrt{\beta} R_{\alpha,\beta-1}. \]

- This would be a good example for a bivariate spectral analysis of growing graphs.
- The next step is to derive a bivariate extension of quantum decomposition.
- *Life is short, but there is always time enough for mathematics!*

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Yichang, China, 2019.08.20–24