relative equilibrium states and random dynamical systems

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Motivation

Definition of RSFT can be intimidating. Before definition, let's start with a motivating example.
Consider a cellular automaton

$$\tau : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$$

Consider the fibers

$$E_y := \tau^{-1}(y), \quad y \in A^{\mathbb{Z}^d}$$

Each fiber E_y is a subset of the full shift $A^{\mathbb{Z}^d}$ characterized by forbidden patterns on finite windows $\{-M, , \cdots , M\}^d + v, v \in \mathbb{Z}^d$, so E_y is like a subshift of finite type (SFT), except that the set of forbidden patterns is not constant and depend on the window location v.
The forbidden patterns characterizing E_y vary with location v according to a dynamical rule:
There's a dynamical system $(Y, \{T^v\}_{v \in \mathbb{Z}^d})$ and a function F defined on Y such that the set of forbidden patterns for location v is $F(T^v y)$. (In fact, $Y = A^{\mathbb{Z}^d}$)

Motivating example

So we got a collection of SFT-like objects E_y, indexed by points of a dynamical system.
This is an example of an RSFT.

For another motivating example, suppose we have a subshift $X \subset A^{\mathbb{Z}^d}$, not necessarily finite type.
Suppose X has a factor $\pi : X \to Y$ such that each fiber $\pi^{-1}(y)$ is of finite type like E_y in the previous slide.
Again, we have a collection of SFT-like objects $\pi^{-1}(y)$ indexed by points of a dynamical system, namely, Y. This is nice because:
Dynamical questions about the original subshift X may be answered by combining results about Y and results about $\{\pi^{-1}(y) : y \in Y\}$.

Definition

A collection $\{E_\omega\}_{\omega\in\Omega}$ is a (one-dimensional, one-step) random subshift of finite type or RSFT if

- it is indexed by points of a measure preserving system $(\Omega, \mathbb{P}, \theta)$ and
- $E_\omega \subset \{1, \cdots, \ell\}^{\mathbb{Z}}$ and
- there exists a measurable map $\Omega \ni \omega \mapsto A_\omega \in \{0,1\}^{\ell\times\ell}$ (random 0-1 matrix) s.t. for all $x = (x_n)_n \in \{1, \cdots, \ell\}^{\mathbb{Z}}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$x \in E_\omega \iff (\forall n)\ A_{\theta^n(\omega)}(x_n, x_{n+1}) = 1$$

In other words, each E_ω is like an SFT defined by the sequence of matrices $(A_{\theta^n(\omega)})_{n\in\mathbb{Z}}$ instead of one matrix.
We may assume

- the base system $(\Omega, \mathbb{P}, \theta)$ is ergodic.
- E_ω is non-empty for $\mathbb{P}$-a.e. ω.

Examples

If $A_\omega = A \in \{0,1\}^{\ell\times\ell}$ (constant case), then E_ω reduces to the classical SFT defined by matrix A.
If each A_ω is a permutation matrix, then E_ω changes with ω, but it always has constant size l.

Given a factor map π from an SFT X to Y, we can associate an RSFT in the following way:

- (WLOG) π is from a 1-block factor map $\pi_0 : \{1,\cdots,\ell\} \to \{1,\cdots,\ell'\}$ and X is from a binary matrix $A \in \{0,1\}^{\ell\times\ell}$
- Define $A_y := \pi_0^{-1}(y_0)|A|\pi_0^{-1}(y_1)$
- Observe that each fiber $\pi^{-1}(y)$ is the subset of $\{1,\cdots,\ell\}^{\mathbb{Z}}$ constrained by $(A_{\sigma^n(y)})_{n\in\mathbb{Z}}$
- E_ω is exactly $\pi^{-1}(y)$
- Given any ergodic measure ν for Y, we can set $(\Omega,\mathbb{P},\theta) := (Y,\sigma_Y,\nu)$. (Choosing an ergodic measure on Y is necessary in the first two slides as well.)

RSFT as factor map

Conversely, given an RSFT $((\Omega, \mathbb{P}, \theta), A : \Omega \to \{0,1\}^{\ell \times \ell})$, we can associate a factor map from an SFT in the following way:

- Define $(Y, \nu) := ((\{0,1\}^{\ell \times \ell})^{\mathbb{Z}}, \mathbb{P}^*)$
- Define SFT $X \subset (\{0,1\}^{\ell \times \ell} \times \{1, \cdots, \ell\})^{\mathbb{Z}}$ with the following rule: (the letter $(A_i, x_i) \in \{0,1\}^{\ell \times \ell} \times \{1, \cdots, \ell\}$ can follow (A_{i+1}, x_{i+1}) if $A_i(x_i, x_{i+1}) = 1$).
- Let $\pi : X \to Y$ be the projection map.
- Now the fiber $\pi^{-1}(y)$ is the same thing as $\{y\} \times E_y$

So, giving an RSFT $((\Omega, \mathbb{P}, \theta), A : \Omega \to \{0,1\}^{\ell \times \ell})$ is the same as giving a factor map $\pi : X \to Y$ from an SFT and an ergodic measure ν on Y. Up to ν-null set of fibers.

Further correspondences

Topological entropy of the RSFT E_ω = the relative topological entropy of fibers $\pi^{-1}(y)$.
Giving a probability measure μ_ω on E_ω for $\mathbb{P}$-a.e. $\omega \in \Omega$ is the same thing as giving a probability measure μ_y on $\pi^{-1}(y)$ for ν-a.e. $y \in Y$.

Form the disjoint union $E = \bigcup_{\omega\in\Omega}\{\omega\} \times E_\omega \subset \Omega \times \{1, \cdots, \ell\}^{\mathbb{Z}}$ and define a skew product transformation $\Theta : E \to E,\ \Theta(\omega, x) := (\theta(\omega), \sigma(x))$.
(Caution: We can't call Θ a measure preserving transformation because we didn't specify a measure on E. It's not a topological dynamical system because we didn't specify a topology on E.)
Then the transformation $\Theta : E \to E$ corresponds to the transformation $\sigma_X : X \to X$. (A precise statement of this is that after discarding some P-null set from E and some ν-null set of fibers from X, there is a measurable conjugacy between two transformations such that its restriction to each fiber is a homeomorphism $\{\omega\} \times E_\omega \to \pi^{-1}(y)$.)

Further correspondences

Giving an invariant measure μ for $\Theta : E \to E$ such that it projects to $\mathbb{P}$ is the same thing as giving an invariant measure μ for the SFT $\sigma_X : X \to X$ such that it projects to ν.

Above is the same thing as giving μ_ω on E_ω for $\mathbb{P}$-a.e. $\omega \in \Omega$ such that $\omega \mapsto \mu_\omega$ is measurable and equivariant.

Such measure μ is called an invariant measure of the RSFT.

RSFT analogues of classical results on SFT

There's always an invariant measure μ of a given RSFT.

The RSFT variational principle holds:
The topological entropy of RSFT E_ω is the supremum of the (relative) entropies of invariant measures μ.

Theorem (Gundlach and Kifer 2000): If the RSFT is topologically mixing, measure of maximal entropy (MME) is unique.
An RSFT is topologically mixing if for a.e. $\omega \in \Omega$ there is a length $L(\omega) \in \mathbb{N}$ such that the product

$$A_\omega A_{\theta\omega} \cdots A_{\theta^{L(\omega)}\omega}$$

is positive (or subpositive).

Non-mixing RSFTs

RSFTs from the first two slides are usually not mixing.
Natural question: Can we write any RSFT E_ω as a disjoint union of finitely many mixing RSFT $E_{1,\omega}, E_{2,\omega}, \cdots, E_{d,\omega}$?

Quick answer: Not always possible. There are at least two obstructions:

- (reducible SFT) Let $A_\omega := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
- (multiplicity) i.i.d. of permutation matrices

Review of relevant facts on SFT

- Recall that given a SFT X, the nonwandering part $X' \subset X$ is another SFT and X' is a disjoint union of finitely many irreducible components.
- Each irreducible component unwinds to a mixing SFT.
- Every invariant measure on X corresponds to an invariant measure on one of these finitely many mixing SFTs.

Better question: Can we do something like above for any RSFT?

Some progress

Theorem (Allahbakhshi and Quas 2012). Given $\pi : X \to Y$ and ν on Y, recall that this is same as giving an RSFT, then there is a finite number c, called class degree, such that for ν-a.e. $y \in Y$, the fiber $\pi^{-1}(y)$ is a disjoint union of finitely many transition classes and there are exactly c of them.

Natural question: Does this answer the previous question? Not sufficient. Promising aspects of this theorem:

- If the class degree c is one, then the RSFT is mixing. And vice versa.
- Number of MME of the RSFT is bounded by c.

So it's natural to expect that the RSFT should be a disjoint union of c mixing RSFTs and that each of these RSFTs is a transition class.
But the transition classes usually do not form an RSFT, let alone a mixing RSFT. They may not even be closed sets.

Further progress

Theorem (Allahbakhshi, Hong, Jung 2014). Given $\pi : X \to Y$ and ν on Y, and if X is irreducible and ν has full support, then the transition classes are closed sets.

But it's not always the case that a given RSFT corresponds to the above setting.
Even if we are given an RSFT satisfying the above, transition classes may not form an RSFT.

Conjecture: For any RSFT $\{E_\omega\}_{\omega\in\Omega}$, there is a sub-RSFT $\{E'_\omega\}_{\omega\in\Omega}$ satisfying the above condition, and every invariant measure of the RSFT $\{E_\omega\}_{\omega\in\Omega}$ lives inside the sub-RSFT.
I believe this sub-RSFT should be called the nonwandering part.

Some partial result

We can sometimes unwind the transition classes into mixing RSFTs, in the following way. (in draft)

Suppose we are given $\pi : X \to Y$ and ν on Y. Assume X is irreducible and ν has full support so that transition classes are at least closed, even though they may not be RSFTs.

- for ν-a.e. $y \in Y$, define a map $F_y : \pi^{-1}(y) \to Z$ collapsing each transition class into a single point. We will denote its image as $Z_y \subset Z$. (There are many ways to do this. For example, we can set $Z_y := \{1, \cdots, c\}$ or $Z_y := \{\text{all transition classes in } \pi^{-1}(y)\}$. The only thing that matters is that $y \mapsto F_y$ is measurable and that there's a permutation $Z_y \to Z_{\sigma(y)}$ induced by the map $\pi^{-1}(y) \to \pi^{-1}(\sigma(y))$.)
- there is a maximal invariant partition $Z_y = \bigcup_{i=1}^{c'} Z_{i,y}$. Here $c' \leq c$
- for each $1 \leq i \leq c'$, the disjoint union $Z_i := \bigcup_{y \in Y} \{y\} \times Z_{i,y}$ is an ergodic dynamical system.
- Transition classes indexed by points of Z_i form a mixing RSFT.
- In short, there are c transition classes and they unwind to c' mixing RSFTs.

Thanks

Thank you!