On decomposition of factor maps between shift spaces on groups - $\mathbb{Z}$ to countable amenable groups

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Shift spaces and codes

- The *full $\mathcal{A}$-shift* $\mathcal{A}^{\mathbb{Z}}$ is the set of all bi-infinite sequences over a finite set $\mathcal{A}$.
- The *shift map* σ on $\mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma(x)_i = x_{i+1}$. A *shift space*, or a *subshift* is a *σ-invariant closed* subset of a full shift.
- A *sliding block code* (simply, a *code*) is a σ-commuting continuous map between shift spaces;
  ϕ is a *factor* code (*conjugacy*) if it is *surjective* (*bijective*).
- A *topological Markov chain* determined by an $r \times r$, 0-1 matrix A is the set of all $x = (x_i) \in \{1, \ldots, r\}^{\mathbb{Z}}$ with $A_{x_i x_{i+1}} = 1$ for $i \in \mathbb{Z}$.
- A subshift is an *SFT* if it is conjugate to a topological Markov chain.
- A *sofic shift* is an image of an SFT under a code.

Decomposition of codes

- A *decomposition* of a code ϕ is a tuple $(\phi_1, \cdots, \phi_n)$ of codes between shift spaces such that $\phi = \phi_n \cdots \phi_1$.

Theorem (Williams, Nasu)

Every conjugacy between two shift spaces is the composition of simple elementary conjugacies, namely, splitting codes and amalgamation codes.

Decompositions of codes arise in *automorphism groups of SFTs*, (eventual) *factor theorems*, *lifting factor maps to closing maps*, and *construction of SFTs between two shifts*, and so on.

Decomposition Problems

Question (Adler and Marcus)

Can any factor code between irreducible SFTs with the same entropy be represented as a composition of *closing* codes?

answered *negatively* by Kitchens. However,

Theorem (Kitchens, Marcus and Trow)

Let $\phi : X \to Y$ be a factor code between irreducible SFTs with the same entropy. Then for all large $n \in \mathbb{N}$, the code $\phi : X^n \to Y^n$ is a composition of closing codes.

Theorem (Kitchens, Marcus and Trow; Boyle)

Let $\phi : X \to Y$ be a factor code between irreducible SFTs with the same entropy. Then there is a factor code $\psi : Z \to X$ such that $\phi \circ \psi$ is a composition of closing codes.

Decomposition Problems

Question (Adler and Marcus)

Can any factor code between irreducible SFTs with the same entropy be represented as a composition of *closing* codes?

Question (Trow)

Can any factor code between irreducible SFTs with the same entropy be decomposed only in (essentially) finitely many different ways?

Theorem (Boyle)

Let ϕ be a factor code between irreducible SFTs with the same entropy. Then the number of conjugacy classes of decompositions of ϕ is finite.

Question: What happens if the entropies are *different*?

Lindenstrauss' Theorem

Question: What happens if the entropies are *different*?

Theorem (Lindenstrauss)

Let $\phi : (X, T) \to (Y, S)$ *be a factor between topological dynamical systems with* X *and* Y *of finite dimension. Let* $h \in [h(S), h(T)]$.

Then there are a system (Z, U) *and factors* $\phi_1 : (X, T) \to (Z, U)$ *and* $\phi_2 : (Z, U) \to (Y, S)$ *such that* $\phi = \phi_2 \circ \phi_1$ *and* $h(U) = h$.

Even when X and Y are shifts of finite type, constructed Z is far from a subshift. We want to find intermediate systems as "*subshifts*" rather than just topological dynamical systems.

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The set of the entropies of intermediate shifts

Let $\phi : X \to Y$ be a factor code between subshifts with $h(X) > h(Y)$.

- Let $\mathcal{S}(\phi) = \{h(\phi_1(X)) : \phi = \phi_2\phi_1$ with ϕ_1, ϕ_2 factor codes $\}$.
- Let $\mathcal{S}_0(\phi) = \{h(\phi_1(X)) : \phi = \phi_2\phi_1$ with ϕ_1, ϕ_2 factors with $\phi_1(X)$ SFT$\}$.

Proposition (Boyle and Tuncel)

Let X and Y be irreducible SFTs. Then every element in $\mathcal{S}_0(\phi) \setminus \{h(Y)\}$ is a limit point of $\mathcal{S}_0(\phi)$.

Corollary

Let X and Y be irreducible SFTs. Then the number of conjugacy classes of decompositions is infinite.

Proposition (Boyle and Tuncel)

Let X and Y be irreducible SFTs. Then every element in $\mathcal{S}_0(\phi) \setminus \{h(Y)\}$ is a limit point of $\mathcal{S}_0(\phi)$.

Theorem (Hong, J. and Lee)

Let X be an SFT. Then $\mathcal{S}_0(\phi)$ is dense in $[h(Y), h(X)]$.

Density result holds for any $\mathbb{Z}$-subshift

Theorem (J, McGoff and Pavlov)

Let $\phi : X \to Y$ be a factor code. Then $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.

Corollary (J, McGoff and Pavlov)

Let X be a $\mathbb{Z}$-subshift. Then the set of the entropies of the subshift factors of X is dense in $[0, h(X)]$.

Corollary (J, McGoff and Pavlov)

Let X be a $\mathbb{Z}$-subshift. Then the set of the entropies of the 0-dimensional TDS factors of X is precisely $[0, h(X)]$.

- Given $h \in (0, h(X))$, one can construct a chain of subshifts $\{X_i\}_{i\in\mathbb{Z}}$ with $h(X_i) \to h$ such that $X \to X_{i+1} \to X_i$ is a decomposition of $X \to X_i$.
- The inverse limit of the system $\{X_i\}_{i\in\mathbb{Z}}$ is a 0-dimensional TDS factor with entropy h.

Main theorem and the Marker Lemma

Theorem (J, McGoff and Pavlov)

Let $\phi : X \to Y$ be a factor code. Then $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.

The following *Marker Lemma* is an essential ingredient of the proof.

Lemma (Krieger)

Let X be a subshift and $N \geq 1$. Then there is a clopen set $F \subset X$ such that

1. *$\sigma^i(F)$, $0 \leq i < N$, are disjoint.*
2. *if $\sigma^i(x) \notin F$ for $-N < i < N$, then $x_{[-N,N]}$ is p-periodic for some $p < N$.*

One can 'mark' the i-coordinate of x if $\sigma^i(x) \in F$. If two marked coordinates are far, then the intermediate part looks like a periodic point.

Sketch of the proof

Theorem (J, McGoff and Pavlov)

Let $\phi : X \to Y$ be a factor code. Then $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.

- Assume ϕ is 1-block and $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$. Take a large $N \in \mathbb{N}$ and choose a clopen "marker" set $F \subset X$.
- Define 1-block codes $f_{N,k} : X \to (\mathcal{A}_X \cup \mathcal{A}_Y)^{\mathbb{Z}}$ by

$$f_{N,k}(x)_0 = \begin{cases} \phi(x_0) & \text{if } \sigma^j(x) \in F \text{ for some } 0 \leq j < k \\ x_0 & \text{otherwise} \end{cases}$$

  and with the commuting property. Intuitively, $f_{N,k}$ applies ϕ to the k-letters to the left of every occurrence of a marker coordinate of x.

Sketch of the proof, II

Theorem (J, McGoff and Pavlov)

Let $\phi : X \to Y$ be a factor code. Then $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.

- Let Z_k be the subshift consisting of all points over $\mathcal{A}_X \cup \mathcal{A}_Y$ such that no word of the form awb with $a, b \in \mathcal{A}_X$ and $w \in \bigcup_{j=1}^{k} \mathcal{A}_Y^j$ occur.
- Then $f_{N,k}(X) \subset Z_k$.
- The $\bigcap_{n\in\mathbb{N}} Z_n$ has the nonwandering set contained in $P \cup Y$, where $P = \{x \in X : x$ is periodic with period $< N\}$. So $\lim_n h(f_{N,k}(X)) = h(Y)$.
- The difference between $f_{N,k}(X)$ and $f_{N,k+1}(X)$ is: the letters exactly $k+1$ to the left of each marker symbol are mapped by ϕ. The frequency of such change is less than $1/N$.
- Hence the set of the entropies $h(f_{N,k}(X))$ is $\frac{\log|A|}{N}$-*dense* in $[h(Y), h(X)]$.
- As $N \to \infty$, we are done.

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Density result holds for any G-subshift

Let G be a countable amenable group and X, Y be subshifts on G.

Theorem (J, McGoff and Pavlov)

Let $\phi : X \to Y$ be a factor code. Then $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.

Corollary

Let X be a G-subshift. Then the set of the entropies of the subshift factors of X is dense in $[0, h(X)]$.

Corollary

Let X be a G-subshift. Then the set of the entropies of the 0-dimensional G-TDS factors of X is precisely $[0, h(X)]$.

Shift spaces and sliding block codes over groups

Let $\mathcal{A}$ be a finite set and G be a group. As in $\mathbb{Z}$-case...

- The *full $\mathcal{A}$-shift on G* $\mathcal{A}^G = \{x : G \to \mathcal{A}\}$ consists of all functions (configurations) from G to $\mathcal{A}$.
  - We use both x_g and $x(g)$ for the symbol at position $g \in G$.
- For each $g \in G$, let $\sigma^g : \mathcal{A}^G \to \mathcal{A}^G$ be the map $(\sigma^g(x))_h = x_{g^{-1}h}$.
  The *shift action* on $G \times \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma(g, x)_h = (\sigma^g(x))_h = x_{g^{-1}h}$.
- A *shift space* on G, or a *subshift* is a subset $X \subset \mathcal{A}^G$ which is closed and *σ-invariant* (that is, $\sigma^g(X) \subset X$ for each $g \in G$).
- A *sliding block code* is a σ-commuting continuous map between shift spaces: $\sigma^g \circ \phi = \phi \circ \sigma^g$ for each $g \in G$.

(Countable) amenable groups

Let G be a countable group.

- We say that G is *amenable* if there are finite (Fölner) sets $F_n \subset G$ such that

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0 \text{ for each } g \in G.$$

- For a subshift $X \subset \mathcal{A}^G$ on an amenable group with $\{F_n\}$, the topological entropy can be defined by $h(X) = \lim_n \frac{1}{|F_n|}|\mathcal{B}_{F_n}(X)|$, where $\mathcal{B}_F(X)$ is the set of X-patterns on a set $F \subset G$.
  - Entropy theory goes nicely to subshifts and their factors on an amenable group.
- We say $A \subset G$ is *(K, δ)-invariant* if

$$|\{g \in G : Kg \cap \mathcal{A} \neq \emptyset \text{ and } Kg \cap (G \setminus A) \neq \emptyset\}| < \delta|A|.$$

  - that is, 'K-boundary' of A is δ-portion small.
  - If A is (F_n, δ) small and F_n is Fölner, then we can estimate the number of patterns on A using the topological entropy: $\mathcal{B}_A(X) \sim e^{|A|(h(X)+\epsilon)}$.

(Countable) amenable groups

Let G be a countable amenable group with a Fölner sequence $\{F_n\}$.

- We say $\{T_1, \cdots, T_n\}$ *ϵ-quasi-tile* a group G if $\{e\} \subset T_1 \subset \cdots \subset T_n$ and, for any finite $D \subset G$ there are C_i such that
  1. for fixed i, $\{T_i c : c \in C_i\}$ are ϵ-disjoint
  2. for $i \neq j$, $T_i C_i \cap T_j C_j = \emptyset$
  3. the collection $\{T_i C_i\}$ $(1-\epsilon)$-cover D (i.e., $|D \setminus \bigcup T_i C_i| < \epsilon |D|$)

Theorem (Ornstein and Weiss)

Let $\epsilon > 0$. There is N such that in any amenable group G and for any (K, δ), there are sets $\{T_1, \cdots, T_N\}$ that are (K, δ)-invariant sets and ϵ-quasi-tile G.

Marker Lemma for G

Lemma

Let X be a G-subshift. For any finite sets $S, T \subset G$ and $c \in \mathbb{N}$, there is a clopen set $F \subset X$ such that

1. *For any c distinct elements $s_1, \cdots, s_c \in S$, the sets $\{\sigma^{s_i}(F)\}_{i=1}^{c}$ have empty intersection.*
2. *if $x \notin \bigcup_{i \notin S \cdot S^{-1}} \sigma^i F$, then $x(T)$ has at least c (essentially disjoint) periods in SS^{-1}.*

- Krieger's original marker lemma corresponds to $G = \mathbb{Z}$, $c = 2$, $S = [0, N]$ and $T = [-N, N]$.
- The first part guarantees "low density of visits to the set F", and the second part guarantees "small entropy".

Sketch of the proof

Theorem (J, McGoff and Pavlov)

Let G be a countable amenable group and X, Y be subshifts on G. Let $\phi : X \to Y$ be a factor code. Then $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.

- Let ϵ, δ small, and c large. Find a set $\{S_i\}$ which δ-quasi tiles G and $\min_i |S_i|$ large enough. Let $S = \bigcup S_i$.
- For $G = \{g_1, g_2, g_3, \cdots\}$ and a clopen set F, define 1-block codes $f_{F,k} : X \to (\mathcal{A}_X \cup \mathcal{A}_Y)^{\mathbb{Z}}$ by

$$x \in F \Rightarrow f_{F,n}(x)_g = \begin{cases} \phi(x_g) & \text{if } g \in \{g_1, \cdots, g_n\} \\ x_g & \text{otherwise} \end{cases}$$

- Then one can show that for any F and $n \in \mathbb{N}$,

$$h(f_{F,n}(X)) - h(f_{F,n+1}(X)) \leq \log |A| \cdot \mathcal{D}(F),$$

where $\mathcal{D}(F)$ satisfies $\mathcal{D}(F) \to 0$ as the upper Banach density of F goes 0.

Sketch of the proof

Theorem (J, McGoff and Pavlov)

Let G be a countable amenable group and X, Y be subshifts on G. Let $\phi : X \to Y$ be a factor code. Then $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.

- By the first property of Marker Lemma and using a quasi-tiling of F_k by S_i's, the density of visits to F in F_k can be shown small enough.
- Hence no matter what T is, if $\{S_i\}$ δ-quasi-tiles G and c is large, then we have a small difference in the consecutive factors $h(f_{F,n}(X))$.
- Let X_F be the 'limit' of the system of $f_{F,n}(X)$. The theorem will be done if $h(X_F) < h(Y) + \epsilon$.

Sketch of the proof

Theorem (J, McGoff and Pavlov)

Let G be a countable amenable group and X, Y be subshifts on G. Let $\phi : X \to Y$ be a factor code. Then $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.

- Choose another tilings $\{T_1, \cdots, T_n\}$ which are (SS^{-1}, δ)-invariant and ϵ-quasi tiles G. Letting $T = \bigcup T_i$, we can now fix F from Marker Lemma.
- For each F_n in the Fölner sequence, each X_F pattern w on F_n is determined by the product of
  1. $B(w) = \{g \in F_n : w(g) \in \mathcal{A}_X\}$: small exponential growth by ergodic theorem
  2. $y(F_n)$: the number of choices is $\leq e^{(h(Y)+\epsilon)|F_n|}$.
  3. $w(\{g : w(g) \in \mathcal{A}_X\})$ given $B(w)$: small exponential growth by periodicity and quasi-tiling property of S_i's.
- And one can obtain $h(Y) - h(X_F)$ is small enough.