The Garden of Eden Theorem: from cellular automata to algebraic dynamical systems

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The Garden of Eden Theorem (proved by Moore and Myhill in 1963) is a central result in the theory of cellular automata. It gives a necessary and sufficient condition for a cellular automaton $\tau \colon A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ - with finite alphabet A over the free abelaian group $\mathbb{Z}^d$ of rank d as universe to be surjective. This result was extended to amenable groups by CS-Machi'-Scarabotti in 1999. Bartholdi (2010, 2016) showed that the GOE Theorem only holds for amenable groups. Following an indication of Gromov, who suggested that the GOE Theorem should hold, more generally, for dynamical systems with a suitable "hyperbolic flavour", with Michel Coornaert in 2016 we proved a GOE type theorem for Anosov diffeomorphisms of tori. An algebraic dynamical system is a pair (X, G) where X is a compact abelian group and G is a group acting by continuous automorphisms of X. Given a group G we consider the integer group ring $\mathbb{Z}[G]$ and, for $f \in \mathbb{Z}[G]$, we denote by X_f the Pontryagin dual of the abelain group underlying the ring $\mathbb{Z}[G]/\mathbb{Z}[G]f$ obtained by quotienting $\mathbb{Z}[G]$ by the principal ideal generated by f. Then (X_f, G) is called a principal algebraic dynamical system. Hanfeng Li (in press) proved a beautiful general GOE type theorem for expansive principal algebraic dynamical systems over amenable groups. Later, in a joint paper CS-Coornaert-Li (2019) we introduced a notion of "weak expansivity" for principal algebraic dynamical systems and in this framework, we presented a GOE type theorem for abelain groups. This result covers the harmonic models for transient abelian groups as well as the Laplace models on $\mathbb{Z}^d$ for all $d \geq 2$.