Enumeration on row-increasing tableaux of shape $2 \times n$

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Outline

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Integer partitions

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be a partition of \( n \), i.e.

\[
\lambda_1 + \lambda_2 + \cdots + \lambda_k = n,
\]

where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \).

The Ferrers diagram of \( \lambda \) is a left-justified array of cells with \( \lambda_i \) cells in the \( i \)-th row, for \( 1 \leq i \leq k \).

Figure: The Ferrers diagram of a partition \( \lambda = (6, 3, 1) \vdash 10 \).
A semistandard Young tableau \( (SSYT) \) of shape \( \lambda \) is a filling of the Ferrers diagram of \( \lambda \) with positive integers such that every row is strictly increasing and every column is weakly increasing.

A standard Young tableau \( (SYT) \) of shape \( \lambda \vdash n \) is a filling of the Ferrers diagram of \( \lambda \) with \( \{1, 2, \ldots, n\} \) such that every row and column is strictly increasing.

\[
\begin{array}{cccccc}
2 & 4 & 6 & 7 & 8 & 9 \\
4 & 5 & 6 \\
8 \\
\end{array}
\quad \begin{array}{cccccc}
1 & 3 & 4 & 5 & 8 & 10 \\
2 & 6 & 7 \\
9 \\
\end{array}
\]

Figure: A semi-standard Young tableau of shape \((6, 3, 1)\) and a standard Young tableau of shape \((6, 3, 1)\).
Major index and amajor index of a tablau

A descent of an SSYT $T$ is an integer $i$ such that $i+1$ appears in a lower row of $T$ than $i$. $D(T)$: the descent set of $T$. The major index of $T$ is defined by $\text{maj}(T) = \sum_{i \in D(T)} i$. An ascent of $T$ to be an integer $i$ such that $i+1$ appears in a higher row of $T$ than $i$. $A(T)$: the ascent set of $T$. The amajor index of $T$ is defined by $\text{amaj}(T) = \sum_{i \in A(T)} i$.

\[ \begin{array}{cccc}
1 & 2 & 5 & 10 \\
3 & 4 & 8 &  \\
6 &  \\
7 &  \\
9 & 
\end{array} \quad \begin{array}{cccc}
1 & 2 & 5 & 10 \\
3 & 4 & 8 &  \\
6 &  \\
7 &  \\
9 & 
\end{array} \]

Figure: $T \in \text{SYT}(4, 3, 1, 1, 1)$.

$D(T) = \{2, 5, 6, 8\}$, $\text{maj}(T) = 21$. $A(T) = \{4, 7, 9\}$, $\text{amaj}(T) = 20$. 

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Lemma (Stanley’s $q$-hook length formula)
For any partition $\lambda = \sum_i \lambda_i$ of $n$, we have

$$\sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \frac{q^{b(\lambda)} [n]!}{\prod_{u \in \lambda} h(u)}.$$  \hspace{1cm} (1)

Here $b(\lambda) = \sum_i (i - 1)\lambda_i$.

The famous RSK algorithm is a bijection between permutations of length $n$ and pairs of SYTs of order $n$ of the same shape. Under this bijection, the descent set of a permutation is transferred to the descent set of the corresponding “recording tableau”. Therefore many problems involving the statistic descent or major index of pattern-avoiding permutations can be translated to the study of descent and major index of tableaux.
Standard Young tableaux of shape $2 \times n$

For any positive integer $n$, we have

$$C_q(n) = \sum_{T \in \text{SYT}(2 \times n)} q^{\text{maj}(T)} = \frac{q^n}{[n + 1]} \left[ \begin{array}{c} 2n \\ n \end{array} \right].$$

(2)

Here $[n] = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$, $[n]! = [n][n-1] \cdots [1]$ and $\left[ \begin{array}{c} n \\ m \end{array} \right] = \frac{[n]!}{[m]![n-m]!}$.

For example, when $n = 3$, we have

$$C_q(3) = \frac{q^3}{[3 + 1]} \left[ \begin{array}{c} 6 \\ 3 \end{array} \right] = q^3 + q^5 + q^6 + q^7 + q^9.$$

And there are five SYT of shape $2 \times 3$, with major index 3,6,7,5,9.

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Increasing tableaux

An increasing tableau is an SSYT such that both rows and columns are strictly increasing, and the set of entries is an initial segment of positive integers (if an integer \( i \) appears, positive integers less than \( i \) all appear).

We denote by \( \text{Inc}_k(\lambda) \) the set of increasing tableaux of shape \( \lambda \) with entries are \( \{1, 2, \ldots, n - k\} \).

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
2 & 4 & 5 \\
\end{array} \quad \begin{array}{c|c|c}
1 & 2 & 4 \\
2 & 3 & 5 \\
\end{array} \quad \begin{array}{c|c|c}
1 & 2 & 3 \\
3 & 4 & 5 \\
\end{array} \quad \begin{array}{c|c|c}
1 & 2 & 4 \\
3 & 4 & 5 \\
\end{array} \quad \begin{array}{c|c|c}
1 & 3 & 4 \\
2 & 4 & 5 \\
\end{array}
\]

Figure: There are five increasing tableaux in \( \text{Inc}_1(2 \times 3) \).

Increasing tableau is defined by O. Pechenik who studied increasing tableaux in \( \text{Inc}_k(2 \times n) \), i.e., increasing tableaux of shape \( 2 \times n \), with exactly \( k \) numbers appeared twice.

Pechenik’s result

Theorem (O. Pechenik)
For any positive integer \( n \), and \( 0 \leq k \leq n \) we have

\[
S_q(n, k) = \sum_{T \in \text{Inc}_k(2 \times n)} q^\text{maj}(T) = \frac{q^{n+k(k+1)/2}}{[n+1]} \binom{n-1}{k} \binom{2n-k}{n}.
\]  

For example, when \( n = 3, \, k = 1 \) we have

\[
S_q(3, 1) = \sum_{T \in \text{Inc}_1(2 \times 3)} q^\text{maj}(T) = \frac{q^4}{[3+1]} \binom{3-1}{1} \binom{5}{3} = q^8 + q^7 + q^6 + q^5 + q^4.
\]

Pechenik’s proof involves the cyclic sieving of increasing tableaux. We will show a more conceptual proof using the \( q \)-hook length formula.
A refinement of small Shröder number

Setting $q = 1$, we get the cardinality of $\text{Inc}_k(2 \times n)$:

$$s(n, k) = \frac{1}{n + 1} \binom{n - 1}{k} \binom{2n - k}{n}.$$  (4)

$s(n, k)$ is considered as a refinement of the small Schröder number which counts the following sets:

1. Dissections of a convex $(n + 2)$-gon into $n - k$ regions;
2. SYTs of shape $(n - k, n - k, 1^k)$;

In 1996 Stanley gave a bijection between the above two sets.


Pechenik gave a nice bijection between SYTs of shape $(n - k, n - k, 1^k)$ and increasing tableaux in $\text{Inc}_k(2 \times n)$. 
**Schröder paths**

$s(n, k)$ also counts number of small Schröder $n$-paths with $k$ flat steps.

A **Schröder $n$-path** is a lattice path goes from $(0, 0)$ to $(n, n)$ with steps $(0, 1), (1, 0)$ and $(1, 1)$ and never goes below the diagonal line $y = x$. If there is no $F$ steps on the diagonal line, it is called a **small Schröder path**.

There is an obvious bijection between SSYTs in $\mathrm{RInc}_k(2 \times n)$ and Schröder $n$-paths with $k$ steps: read the numbers $i$ from 1 to $2n - k$ in increasing order, if $i$ appears only in row 1 (2), it corresponds to a $U$ ($D$) step, if $i$ appears in both rows, it corresponds to an $F$ step.

**Motivation**: are there any interesting result for these tableaux that correspond to all Schröder $n$-paths?
Row-increasing tableaux

A row-increasing tableau is an SSYT with strictly increasing rows and weakly increasing columns, and the set of entries is a consecutive segment of positive integers.

We denote by $\text{RInc}_{k}^{m}(\lambda)$ the set of row-increasing tableaux of shape $\lambda$ with set of entries $\{m+1, m+2, \ldots, m+n-k\}$. When $m = 0$, we will just denote $\text{RInc}_{k}^{0}(\lambda)$ as $\text{RInc}_{k}(\lambda)$. It is obvious that $\text{Inc}_{k}(\lambda) \subseteq \text{RInc}_{k}(\lambda)$.

![Figure](image)

Figure: There are 6 row-increasing tableaux in $\text{RInc}_{2}(2 \times 3)$.

It is not hard to show that $\text{RInc}_{k}(2 \times n)$ is counted by

$$r(n, k) = \frac{1}{n-k+1} \binom{2n-k}{k} \binom{2n-2k}{n-k}.$$  

(5)

$r(n, k)$ is considered as a refinement of the large Schröder number.
Our Main Results

We study the statistics \texttt{maj} and \texttt{amaj} of SSYTs in $\text{RInc}_k(2 \times n)$ and get the following results.

\textbf{Theorem}
For any positive integer $n$, and $0 \leq k \leq n$ we have

$$R_q(n, k) = \sum_{T \in \text{RInc}_k(2 \times n)} q^{\texttt{maj}(T)} = \frac{q^{n+k(k-3)/2}}{[n-k+1]} \left[ \begin{array}{c} 2n-k \\ k \end{array} \right] \left[ \begin{array}{c} 2n-2k \\ n-k \end{array} \right]. \quad (6)$$

\textbf{Theorem}
For any positive integer $n$, and $0 \leq k \leq n$ we have

$$\tilde{R}_q(n, k) = \sum_{T \in \text{RInc}_k(2 \times n)} q^{\texttt{amaj}(T)} = \frac{q^{k(k-1)/2}}{[n-k+1]} \left[ \begin{array}{c} 2n-k \\ k \end{array} \right] \left[ \begin{array}{c} 2n-2k \\ n-k \end{array} \right]. \quad (7)$$
1 Definitions and Backgrounds

2 Bijective proof of Pechenik’s result

3 Counting major index for $\text{RI}nc_k(2 \times n)$

4 Counting major index for $\text{RI}nc_k(2 \times n)$

5 Counting major index of Schröder $n$-paths
Bijective proof of Pechenik’s result

**Theorem (O. Pechenik)**
There exists a bijection \( \gamma \) between \( \text{Inc}_k(2 \times n) \) and \( \text{SYT}(n - k, n - k, 1^k) \) which preserves the descent set.

Given \( T \in \text{Inc}_k(2 \times n) \). Let \( A \) be the set of numbers that appear twice. Let \( B \) be the set of numbers that appear in the second row immediately right of an element of \( A \). Let \( \gamma(T) \) be the tableau of shape \( (n - k, n - k) \) formed by deleting all elements of \( A \) from the first row of \( T \) and all elements of \( B \) from the second row of \( T \). It is not hard to prove that \( \gamma \) is a bijection.

E.g., in the following example, we have \( A = \{4, 6, 8\} \) and \( B = \{6, 7, 9\} \).

\[
\begin{array}{ccccccc}
1 & 2 & 4 & 5 & 6 & 8 \\
3 & 4 & 6 & 7 & 8 & 9 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & 8 \\
6 \\
7 \\
9 \\
\end{array}
\]
Why does $\gamma$ preserves the major index?

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For each $i \in D(T)$, there are three cases:

1) $i \notin A$. then in $\gamma(T)$ i is still in row 1 and $i + 1$ is still in row 2, and therefore $i \in D(\gamma(T))$;

2) $i \in A$ and $i \notin B$. In this case $i + 1 \in B$. And in $\gamma(T)$, i is in row 2 and $i + 1$ is in the first column with row index $j$ for some $j \geq 3$, thus $i \in D(\gamma(T))$;

3) $i \in A$ and $i \in B$. Since we also have $i + 1 \in B$, then in $S$, i is in the first column with row index $j$ for some $j \geq 3$, and $i + 1$ is in the first column with row index $j + 1$, thus $i \in D(\gamma(T))$;

Combining the above three cases, we have $D(T) \subseteq D(\gamma(T))$. Similarly we can show that $D(\gamma(T)) \subseteq D(T)$. 
Lemma (Stanley’s $q$-hook length formula)

For any partition $\lambda = \sum_i \lambda_i$ of $n$, we have

$$\sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = \frac{q^{b(\lambda)}[n]!}{\prod_{u \in \lambda} h(u)}. \quad (8)$$

Here $b(\lambda) = \sum_i (i - 1) \lambda_i$.

Applying the above formula we have

$$S_q(n, k) = \sum_{T \in \text{Inc}_k(2 \times n)} q^{\text{maj}(T)} = \sum_{T \in \text{SYT}(n-k,n-k,1^k)} q^{\text{maj}(T)}$$

$$= \frac{q^{n+k(k+1)/2} [2n-k]!}{[n-k]! [n-k-1]! [n+1][n][k]!} = \frac{q^{n+k(k+1)/2}}{[n+1]} \left[\begin{array}{c} 2n-k \\ n \end{array}\right] \left[\begin{array}{c} n-1 \\ k \end{array}\right].$$
Definitions and Backgrounds

Bijective proof of Pechenik’s result

Counting major index for $\text{RInc}_k(2 \times n)$

Counting major index for $\text{RInc}_k(2 \times n)$

Counting major index of Schröder $n$-paths
Theorem
For any positive integer \( n, k \), we have

\[
r(n, k) = s(n, k) + s(n, k - 1). \tag{9}
\]

There is a bijection \( f : \text{RInc}_k(2 \times n) \setminus \text{Inc}_k(2 \times n) \leftrightarrow \text{Inc}_{k-1}(2 \times n) \).

Given \( T \in \text{RInc}_k(2 \times n) \setminus \text{Inc}_k(2 \times n) \), find the minimal integer \( j \) such that \( T_{1,j} = T_{2,j} \), i.e., the \( j \)-th column is the leftmost column of \( T \) with two identical entries. Now we first delete the entry \( T_{2,j} \), then move all the entries on the right of \( T_{2,j} \) one box to the left and set the last entry as \( 2n - k + 1 \), and define the resulting tableau to be \( f(T) \).

\[
T: \begin{array}{cccccc}
1 & 3 & 4 & 5 & 6 \\
2 & 3 & 4 & 6 & 7
\end{array} \quad \mapsto \quad f(T): \begin{array}{cccccc}
1 & 3 & 4 & 5 & 6 \\
2 & 4 & 6 & 7 & 8
\end{array}
\]

Figure: An example of \( f \) with \( T \in \text{RInc}_3(2 \times 5) \setminus \text{Inc}_3(2 \times 5) \) and \( f(T) \in \text{Inc}_2(2 \times 5) \).
Note that $f$ does not preserve the major index. In fact we have

**Theorem**

For any positive integer $n, k$ with $k < n$, we have

$$R_q(n, k) = S_q(n, k) + S_q(n, k-1) + (1 - q^{2n-k})(S_q(n-1, k-1) + S_q(n-1, k-2)).$$

(10)

**Proof:** For all SSYT$s$ in $RInc_k(2 \times n) \setminus Inc_k(2 \times n)$, there are two cases.

1. If $T_{1,n} = T_{2,n}$, the last column of $T$ consist of two identical entries $2n - k$ and $2n - k \notin D(T)$.

   \[
   \begin{array}{cccccc}
   1 & 3 & 4 & 5 & 7 \\
   2 & 3 & 4 & 6 & 7 \\
   \end{array}
   \quad \rightarrow \quad
   \begin{array}{cccccc}
   1 & 3 & 4 & 5 & 7 \\
   2 & 4 & 6 & 7 & 8 \\
   \end{array}
   \]

   We will show that the sum of $q^{\text{maj}(T)}$ over all these tableaux is $S_q(n - 1, k - 1) + S_q(n - 1, k - 2)$. 
1) The $n$-th column is the only column of $T$ with identical entries. In this case the last column of $T$ consist of two identical entries $2n - k$ and $2n - k \notin D(T)$. And the sum of $q^{\text{maj}(T)}$ over these tableaux is $S_q(n - 1, k - 1)$.

2) There is at least one column with identical entries in $T$ besides the $n$-th column. Now let $T'$ be the tableau obtained by deleting the last column from $T$. Then $T' \in \text{RIInc}_{k-1}(2 \times (n - 1)) \setminus \text{Inc}_{k-1}(2 \times (n - 1))$. There are two cases for the last column of $T'$.

   a) If $T'_{1,n-1} \neq T'_{2,n-1}$, then $f(T') \in \text{Inc}_{k-2}(2 \times (n - 1))$ with $f(T)_{1,n-1} < 2n - k - 1$, and $\text{maj}(T) = \text{maj}(T') = \text{maj}(f(T'))$;

   \[
   \begin{array}{cccccc}
   1 & 3 & 4 & 5 & 7 \\
   2 & 3 & 4 & 6 & 7 \\
   \end{array}
   \quad \mapsto \quad
   \begin{array}{cccc}
   1 & 3 & 4 & 5 \\
   2 & 3 & 4 & 6 \\
   \end{array}
   \quad \mapsto \quad
   \begin{array}{cccc}
   1 & 3 & 4 & 5 \\
   2 & 4 & 6 & 7 \\
   \end{array}
   \]
b) If $T'_{1,n-1} = T'_{2,n-1}$, we have $T_{1,n-1} = T_{2,n-1} = 2n - k - 1$.

\[
\begin{array}{cccccc}
1 & 3 & 4 & 6 & 7 \\
2 & 3 & 5 & 6 & 7
\end{array}
\quad \mapsto \quad
\begin{array}{cccc}
1 & 3 & 4 & 6 \\
2 & 3 & 5 & 6
\end{array}
\quad \mapsto \quad
\begin{array}{cccc}
1 & 3 & 4 & 6 \\
2 & 5 & 6 & 7
\end{array}
\]

Since $T_{1,n} = T_{2,n} = 2n - k$, we have $2n - k - 1 \in D(T)$ but $2n - k - 1 \notin D(T')$, and all the other descents of $T'$ are also descents of $T$. Thus

\[
maj(T') = maj(T) - (2n - k - 1). \tag{11}
\]

Moreover when we apply $f$ to $T'$ we have $2n - k - 1 \in D(f(T'))$ but $2n - k - 1 \notin D(T')$, and all the other descents of $T'$ are also descents of $f(T')$. Therefore we have

\[
maj(f(T')) = maj(T') + (2n - k - 1). \tag{12}
\]

Combining (11) and (12) we know that $f(T') \in \text{Inc}_{k-2}(2 \times (n - 1))$ with $f(T)_{1,n-1} = 2n - k - 1$, and $maj(f(T')) = maj(T)$.

Thus the sum of $q^{maj(T)}$ over these tableaux of case a) and b) is $S_q(n-1, k-2)$. 

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2. If $T_{1,n} \neq T_{2,n}$. In this case if we apply $f$ to $T$, $2n - k + 1$ will be added as the last entry in row 2 of $f(T)$ and $D(f(T)) \setminus D(T) = \{2n - k\}$ and $\text{maj}(f(T)) = \text{maj}(T) + 2n - k$. Therefore we know that the sum of $q^{\text{maj}(T)}$ over these talbeaux is

$$S_q(n, k - 1) - q^{2n-k}(S_q(n - 1, k - 1) + S_q(n - 1, k - 2)).$$

Combining the Case 1) and Case 2) we have

$$\sum_{T \in \text{RInc}_k(2 \times n) \setminus \text{Inc}_k(2 \times n)} q^{\text{maj}(T)}$$

$$= S_q(n, k - 1) + (1 - q^{2n-k})(S_q(n - 1, k - 1) + S_q(n - 1, k - 2))$$

and

$$R_q(n, k) = \frac{q^{n+k(k-1)/2}}{\begin{bmatrix} n \\ k \end{bmatrix}} \begin{bmatrix} 2n - k \\ n \end{bmatrix}.$$
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Counting amajor index for $\text{RInc}_k(2 \times n)$

$$\tilde{R}_q(n, k) = \sum_{T \in \text{RInc}_k(2 \times n)} q^{\text{amaj}(T)} = \frac{q^{k(k-1)/2}}{[n-k+1]} \binom{2n-k}{k} \binom{2n-2k}{n-k}.$$ 

We will prove the above formula by showing that 

$$\sum_{T \in \text{RInc}_k(2 \times n)} q^{\text{maj}(T)} = q^{n-k} \cdot \sum_{T \in \text{RInc}_k(2 \times n)} q^{\text{amaj}(T)}.$$ 

For example, there are 6 row-increasing tableaux in $\text{RInc}_2(2 \times 3)$, with the $(\text{maj}, \text{amaj})$ pairs $(4,3),(4,5),(2,4),(5,1),(3,3),(6,2)$.

We want to establish a bijection $\Phi : \text{RInc}_k(2 \times n) \leftrightarrow \text{RInc}_k(2 \times n)$ such that 

$$\text{maj}(\Phi(T)) = \text{amaj}(T) + n - k.$$
The general result

**Theorem**
There is a bijection $\Phi : \text{RInc}_k(2 \times n) \leftrightarrow \text{RInc}_k(2 \times n)$ that preserves the second row, and

$$\text{maj}(\Phi(T)) = \text{amaj}(T) + n - k.$$
The prime case

A row-increasing tableau $T$ is prime if for each integer $j$ satisfies $T_{1,j+1} = T_{2,j} + 1$, $T_{2,j+1}$ also appears in row 1 in $T$.

$pRInc^m_k(\lambda)$: prime row-increasing tableaux of shape $\lambda$ with set of entries $\{m + 1, m + 2, \ldots, m + n - k\}$.

For each $T \in pRInc^m_k(2 \times n)$, let $A$ be the set of numbers that appear twice, and $B$ be the set of numbers that appear in the second row immediately left of an element of $A$ in cyclic order.

Let $g(T)$ be the tableau of shape $2 \times n$ obtained by first deleting all elements in $A$ from the first row and then inserting all elements in $B$ into the first row and list them in increasing order, and keep the entries in row 2 unchanged.

In the following example, we have $A = \{2, 6, 9\}$ and $B = \{3, 8, 9\}$.

\[
T: \begin{array}{cccccc}
1 & 2 & 4 & 5 & 6 & 9 \\
2 & 3 & 6 & 7 & 8 & 9 \\
\end{array}
\quad \xrightarrow{g} \quad g(T): \begin{array}{cccccc}
1 & 3 & 4 & 5 & 8 & 9 \\
2 & 3 & 6 & 7 & 8 & 9 \\
\end{array}
\]
Lemma
The map $g$ is an injection from $\text{pRInc}_k^m(2 \times n)$ to $\text{RInc}_k^m(2 \times n)$ which satisfies the following:

1) If $T_{2,1}$ appears only once in $T$, then $g(T)_{1,i+1} \leq g(T)_{2,i}$ for each $i, 1 \leq i \leq n - 1$;
2) $T_{2,1}$ appears twice in $T$ if and only if $g(T)_{1,n} = g(T)_{2,n}$.

Sketch of Proof: there are two cases:

- $T_{2,1}$ appears only once in $T$;

\[
T: \begin{array}{cccccc}
5 & 7 & 8 & 10 & 11 & 12 \\
6 & 8 & 9 & 12 & 13 & 14 \\
\end{array}
\]

\[\rightarrow\]

\[
g(T): \begin{array}{cccccc}
5 & 6 & 7 & 9 & 10 & 11 \\
6 & 8 & 9 & 12 & 13 & 14 \\
\end{array}
\]
Lemma
The map $g$ is an injection from $\text{pRInc}_k^m(2 \times n)$ to $\text{RInc}_k^m(2 \times n)$ which satisfies the following:

1) If $T_{2,1}$ appears only once in $T$, then $g(T)_{1,i+1} \leq g(T)_{2,i}$ for each $i, 1 \leq i \leq n - 1$;

2) $T_{2,1}$ appears twice in $T$ if and only if $g(T)_{1,n} = g(T)_{2,n}$.

Sketch of Proof: there are two cases:

- $T_{2,1}$ appears twice in $T$;

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<th>$T$:</th>
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- $g$ | $g(T)$: | 1 | 3 | 4 | 5 | 8 | 9 |
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Lemma

For each $T \in \text{pRInc}_k^m(2 \times n)$ we have

$$\text{maj}(g(T)) = \begin{cases} 
\text{amaj}(T) + n - k, & \text{if } T_{1,1} = T_{2,1}; \\
\text{amaj}(T) + m + n - k, & \text{if } T_{1,1} \neq T_{2,1}.
\end{cases}$$

(13)

Proof: let $T^0$ be the skew shape tableau obtained by deleting the numbers in $A$ from row 1 of $T$ and “push” all the remaining numbers to the right, and keep the second row unchanged.

$$
\begin{array}{cccccc}
T: & 5 & 6 & 8 & 9 & 10 & 13 \\
& 7 & 8 & 11 & 12 & 13 & 14 \\
\end{array}
\quad \xrightarrow{g} \quad
\begin{array}{cccccc}
g(T): & 5 & 6 & 7 & 9 & 10 & 12 \\
& 7 & 8 & 11 & 12 & 13 & 14 \\
\end{array}

T^0: 
\begin{array}{cccc}
 & 5 & 6 & 9 & 10 \\
7 & 8 & 11 & 12 & 13 & 14 \\
\end{array}
$$

1. $D(g(T)) \setminus D(T^0) = A(T) \setminus A(T^0)$.

and therefore $\text{maj}(g(T)) - \text{maj}(T^0) = \text{amaj}(T) - \text{amaj}(T^0)$;
2. when $T_{1,1} \neq T_{2,1}$, $\text{maj}(T^0) = \text{amaj}(T^0) + m + n - k$.

Suppose the descents of $(T^0)$ appears in columns $k + x_1, k + x_2, \ldots, k + x_d$ in row 1, and the ascents of $(T^0)$ appears in columns $y_1, y_2, \ldots, y_{d-1}$ in row 2. Here $d, x_1, \ldots, x_d, y_1, \ldots y_{d-1}$ are all positive integers and $x_d = n - k$. It is not hard to check that $T^0$ is determined by the set $X = \{x_1, x_2, \ldots, x_d\} \subseteq$ and $Y = \{y_1, y_2, \ldots, y_{d-1}\} \subseteq$. And we have

$$D(T^0) = \{m + x_1, m + x_2 + y_1, \ldots, m + x_d + y_{d-1}\};$$
$$A(T^0) = \{m + x_1 + y_1, m + x_2 + y_2, \ldots, m + x_{d-1} + y_{d-1}\}.$$

Therefore we have $\text{maj}(T^0) - \text{amaj}(T^0) = m + x_d = m + n - k$.

An example for $m = 4$, $n = 6$, $k = 2$, $d = 2$, $X = \{2, 4\}$ and $Y = \{2\}$.

$$T^0:\begin{array}{|c|c|c|c|c|c|}
\hline
 & 5 & 6 & 9 & 10 \\
\hline
7 & 8 & 11 & 12 & 13 & 14 \\
\hline
\end{array}$$

Similarly we can prove that when $T_{1,1} = T_{2,1}$,

$\text{maj}(T^0) = \text{amaj}(T^0) + n - k$. 
The general case

Given $T \in \text{RInc}_k(2 \times n)$, we can uniquely decompose $T$ into prime row-increasing tableaux $T_1 T_2 \cdots T_l$, and set $\Phi(T) = g(T_1)g(T_2)\cdots g(T_l)$.

An example with $n = 13$, $k = 6$, and $l = 3$. Here

$A(T) = \{3, 8, 9, 11, 13, 15, 17, 19\}$, $A(T^0_1) = \{3\}$, $A(T^0_2) = \{11, 13\}$, $A(T^0_3) = \emptyset$, $D(T^0_1) = \{1, 5\}$, $D(T^0_2) = \{10, 12, 14\}$, $D(T^0_3) = \{18\}$. $D(\Phi(T)) = \{1, 5, 8, 10, 12, 14, 15, 18, 19\}$. $\text{amaj}(T) = 95$ and $\text{maj}(\Phi(T)) = 102$.

$T$:

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$\Phi(T)$:

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Proof for the general case (sketch)

Suppose $T_j \in \text{pRInc}_{k_j}^{m_j}(2 \times n_j)$ for integers $m_j, k_j, n_j$ with $m_j, k_j \geq 0$, and $n_j > 0$. We have

$$n_1 + n_2 + \cdots + n_l = n, \quad k_1 + k_2 + \cdots + k_l = k.$$ 

The smallest entry of $T_i$ is $m_i + 1$ with $m_1 = 0$, and

$$m_j = 2(n_1 + n_2 + \cdots + n_{j-1}) - (k_1 + k_2 + \cdots + k_{j-1}), \quad 2 \leq j \leq l,$$

And the the largest entry of $T_j$ is $m_{j+1}$ for each $j, 1 \leq j \leq l - 1$,

It is easy to check that

$$A(T) = A(T_1) \cup A(T_2) \cup \cdots \cup A(T_l) \cup \{m_2, m_3, \ldots, m_l\}$$

and

$$\text{amaj}(T) = \sum_{j=1}^{l} \text{amaj}(T_j) + \sum_{j=2}^{l} m_j.$$
Proof for the general case (sketch)

Moreover we have

\[ D(\Phi(T)) = D(g(T_1)) \cup D(g(T_2)) \cup \cdots \cup D(g(T_l)). \]

Since for each \( j, 1 \leq j \leq l, \)

\[ \text{maj}(g(T_j)) = \text{amaj}(T_j) + m_j + n_j - k_j, \]

Therefore we have

\[
\text{maj}(\Phi(T)) = \sum_{j=1}^{l} \text{maj}(g(T_j)) = \sum_{j=1}^{l} (\text{amaj}(T_j) + m_j + n_j - k_j)
\]

\[
= \sum_{j=1}^{l} \text{amaj}(T_j) + \sum_{j=1}^{l} m_j + n - k
\]

\[ = \text{amaj}(T) + n - k. \]
1 Definitions and Backgrounds

2 Bijective proof of Pechenik’s result

3 Counting major index for $\text{RInc}_k(2 \times n)$

4 Counting major index for $\text{RInc}_k(2 \times n)$

5 Counting major index of Schröder $n$-paths
Counting major index of Schröder $n$-paths

Let $P$ be a Schröder $n$-path that goes from the origin $(0,0)$ to $(n,n)$ with $k$ $F$ steps, we can associate with $P$ a word $w = w(P) = w_1w_2 \ldots w_{2n-k}$ over the alphabet $\{0,1,2\}$ with exactly $k$ 1's.

**Figure:** A Schröder $P$ with $\omega(P) = 00100021222022$. 
Counting major index of Schröder $n$-paths

The descent set of $w$ is the set of all positions of the descents of $w$, $D(w) = \{i : 1 \leq i \leq n, w_i > w_{i+1}\}$. The major index of $w$ is defined as $\text{maj}(w) = \sum_{i \in D(w)} i$. And define $\text{maj}(P) = \text{maj}(w(P))$.

In 1993, Bonin, Shapiro and Simion study the major index for Schröder paths and gave the following result:

$$\sum q^{\text{maj}(P)} = \frac{1}{[n-k+1]} \left[\begin{array}{c} 2n-k \\ k \end{array} \right] \left[\begin{array}{c} 2n-2k \\ n-k \end{array} \right].$$

(14)

Here the sum is over all Schröder $n$-paths with exactly $k$ $F$ steps.

An obvious bijection between SSYT's in $\text{RInc}_k(2 \times n)$ and Schröder $n$-paths with $k$ steps: read the numbers $i$ from 1 to $2n-k$ in increasing order, if $i$ appears only in row 1 (2), it corresponds to a $U (D)$ step, if $i$ appears in both rows, it corresponds to an $F$ step.

A naive thinking is that if the $i$-th step corresponds to a descent in $P$, then $i$ is an ascent of $T$, i.e., $D(P)=A(T)$ and $\text{maj}(P) = \text{amaj}(T)$.

But this is NOT true.
Thank you!