On Pronormal Subgroups of Finite Groups

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Agreement. Further we consider finite groups only.

**Definition (Ph. Hall).**

A subgroup $H$ of a group $G$ is *pronormal* in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

**Theorem* (Ph. Hall, 1960s).** Let $G$ be a group and $H \leq G$. The following conditions are equivalent:

1. $H$ is pronormal in $G$;
2. In any transitive permutation representation of $G$, the subgroup $N_G(H)$ acts transitively on the set $\text{fix}(H)$.

**Examples.** The following subgroups are pronormal in finite groups:

- Normal subgroups;
- Maximal subgroups;
- Sylow subgroups.
Definitions and Examples

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Let a group $G$ acts transitively on a set $\Omega$.

Define an equivalence relation $\rho$ on $\Omega$ by the following way: $x \rho y$ if and only if $G_x = G_y$.

Let $\Omega = \bigcup_{x \in \Omega} \Delta(x)$ be a partition of $\Omega$.

Proposition. Let $x \in \Omega$. The following conditions are equivalent:
(1) $G_x$ is pronormal in $G$;
(2) for each $y \in \Omega$ there exists $t \in \langle G_x, G_y \rangle$ s. t. that $\Delta(x)^t = \Delta(y)$. 
Pronormality works...

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

**Definition (L. Babai).** A group $G$ is called a *CI-group* if between every two isomorphic relational structures on $G$ (as underlying set) which are invariant under the group $G_R = \{ g_R \mid g \in G \}$ of right multiplications

$$g_R : x \mapsto xg,$$

there exists an isomorphism which is at the same time an automorphism of $G$.

**Theorem (L. Babai, 1977).** $G$ is a CI-group if and only if $G_R$ is pronormal in $\text{Sym}(G)$.

**Corollary.** If $G$ is a CI-group then $G$ is abelian.

**Theorem (P. Pálfy, 1987).** $G$ is a CI-group if and only if $|G| = 4$ or $G$ is cyclic of order $n$ such that $(n, \varphi(n)) = 1$. 
$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

**General Problem.** Given a group $G$ and $H \leq G$. Is $H$ pronormal in $G$?
Properties of Pronormal Subgroups

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Proposition (The Frattini Argument). Let $A \unlhd G$ and $H \leq A$. The following statements are equivalent:

1. $H$ is pronormal in $G$;
2. $H$ is pronormal in $A$ and $G = AN_G(H)$.

Proposition. Let $A \trianglelefteq G$ and $H \leq G$. The following statements are equivalent:

1. $H$ is pronormal in $G$;
2. $HA/A$ is pronormal in $G/A$ and $H$ is pronormal in $N_G(HA)$. 

General Problem: Reductions

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

General Problem. Given a group $G$ and $H \leq G$. Is $H$ pronormal in $G$?

Assume that $G$ is not simple and $A$ is a minimal normal subgroup of $G$. Then $A$ is a direct product of simple groups and one of the following cases arises:

(1) If $A \leq H$, then $H$ is pronormal in $G$ if and only if $HA/A$ is pronormal in $G/A$. Note that $|G/A| < |G|$.

(2) If $H \leq A$, then $H$ is pronormal in $G$ if and only if $H$ is pronormal in $A$ and $G = AN_G(H)$. We need to know pronormal subgroups in direct products of simple groups.

(3) If $H \not\leq A$ and $A \not\leq H$, then $H$ is pronormal in $G$ if and only if $N_G(HA) = AN_{N_G(HA)}(H)$ and $H$ is pronormal in $HA$. We need to find good restrictions to $G$ and $H$. 
Overgroups of Pronormal Subgroups

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Theorem (Ch. Praeger, 1984). Let $G$ be a transitive permutation group on a set $\Omega$ of $n$ points, and let $K$ be a nontrivial pronormal subgroup of $G$. Then

(a) $|\text{fix}(K)| \leq \frac{1}{2}(n - 1)$, and

(b) if $|\text{fix}(K)| = \frac{1}{2}(n - 1)$ then $K$ is transitive on its support in $\Omega$, and either $G \geq A_n$, or $G = GL(d, 2)$ acting on the $n = 2^d - 1$ nonzero vectors, and $K$ is the pointwise stabilizer of a hyperplane.

Remark. It is interesting to check the pronormality of overgroups of pronormal (in particular, Sylow) subgroups.
Theorem* (Ph. Hall, 1960s). Let $G$ be a group and $H \leq G$. The following conditions are equivalent:

(1) $H$ is pronormal in $G$;
(2) In any transitive permutation representation of $G$, the subgroup $N_G(H)$ acts transitively on the set $fix(H)$.

Corollary*. Let $G$ be a group, $S \leq H \leq G$ and $S$ be a pronormal (for example, Sylow) subgroup of $G$. Then the following conditions are equivalent:

(1) $H$ is pronormal in $G$;
(2) $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in N_G(S)$.

Lemma 1 (A. Kondrat’ev, 2005). Let $G$ be a nonabelian simple group and $S \in Syl_2(G)$. Then either $N_G(S) = S$ or $(G, N_G(S))$ is known.

Conjecture (E. Vdovin and D. Revin, 2012). The subgroups of odd index (= the overgroups of Sylow 2-subgroups) are pronormal in simple groups.
Subgroups of Odd Index

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Let $G = AH$, where $A$ is a minimal normal subgroup of $G$ and $H$ is a subgroup of odd index in $G$. If $A$ is of odd order, then $A$ is abelian and we can use the following assertion.

Theorem 1 (A. Kondrat’ev, N.M., and D. Revin, 2016). Let $H$ and $V$ be subgroups of a group $G$ such that $V$ is an abelian normal subgroup of $G$ and $G = HV$. Then the following statements are equivalent:

(1) $H$ is pronormal in $G$;
(2) $U = N_U(H)[H,U]$ for any $H$-invariant subgroup $U \leq V$.

If $A$ is a minimal normal subgroup, then $H$ is pronormal in $G = AH$. 
Subgroups of Odd Index

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Let $G = AH$, where $A$ is a minimal normal subgroup of $G$ and $H$ is a subgroup of odd index in $G$.

If $A$ is of even order, then $A$ is a nonabelian simple group and in some cases we can use the following assertion.

Theorem 2 (A. Kondrat’ev, N.M., and D. Revin, 2017). Let $G$ be a group, $A \leq G$, the overgroups of Sylow $p$-subgroups are pronormal in $A$, and $T \in Syl_p(A)$. Then the following statements are equivalent:

1. the overgroups of Sylow $p$-subgroups are pronormal in $G$;
2. the overgroups of Sylow $p$-subgroups are pronormal in $N_G(T)/T$ and for each $H \leq G$ if the index $|G : H|$ is not divisible by $p$, then $N_G(H)A/A = N_{G/A}(HA/A)$.

We need to know pronormality of subgroups of odd index in simple groups and in direct products of simple groups.
$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Conjecture (E. Vdovin and D. Revin, 2012). The subgroups of odd index (= the overgroups of Sylow 2-subgroups) are pronormal in simple groups.

Corollary*. Let $G$ be a group, $S \leq H \leq G$ and $S$ be a pronormal (for example, Sylow) subgroup of $G$. Then the following conditions are equivalent:
(1) $H$ is pronormal in $G$;
(2) $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in N_G(S)$.

Remark. Let $G$ be a group, $H \leq G$ and $S$ be a pronormal subgroup of $G$. If $N_G(S) \leq H$ then $H$ is pronormal in $G$. 
A group \( G \) is simple if \( G \) does not contain proper normal subgroups.

With respect to the *Classification of Finite Simple Groups*, finite simple groups are:

- Cyclic groups \( C_p \), where \( p \) is a prime;
- Alternating groups \( Alt(n) \) for \( n \geq 5 \);
- Classical groups: \( PSL_n(q) = L_n(q) \), \( PSU_n(q) = U_n(q) = PSL_n^-(q) = L_n^-(q) \), \( PSp_{2n}(q) = S_{2n}(q) \), \( P\Omega_n(q) = O_n(q) \) (\( n \) is odd), \( P\Omega_n^+(q) = O_n^+(q) \) (\( n \) is even), \( P\Omega_n^-(q) = O_n^-(q) \) (\( n \) is even);
- Exceptional groups of Lie type:
  - \( E_8(q) \), \( E_7(q) \),
  - \( E_6(q) \), \( 2E_6(q) = E_6^-(q) \),
  - \( 3D_4(q) \), \( F_4(q) \), \( 2F_4(q) \),
  - \( G_2(q) \), \( 2G_2(q) = Re(q) \) (\( q \) is a power of 3),
  - \( 2B_2(q) = Sz(q) \) (\( q \) is a power of 2);
- 26 sporadic groups.
Normalizers of Sylow 2-subgroups

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

If $G$ is a group, $S$ is a Sylow subgroup of $G$, and $N_G(S) \leq H$, then $H$ is pronormal in $G$.

Lemma 1 (A. Kondrat’ev, 2005). Let $G$ be a nonabelian simple group and $S \in Syl_2(G)$. Then $N_G(S) = S$ excluding the following cases:

1. $G \cong J_2$, $J_3$, $Suz$ or $HN$;
2. $G \cong 2G_2(3^{2n+1})$ or $J_1$;
3. $G$ is a group of Lie type over field of characteristic 2;
4. $G \cong PSL_2(q)$, where $3 < q \equiv \pm 3 \pmod{8}$;
5. $G \cong PSp_{2n}(q)$, where $n \geq 2$ and $q \equiv \pm 3 \pmod{8}$;
6. $G \cong PSL^n_\eta(q)$, where $n \geq 3$, $\eta = \pm$, $q$ is odd, and $n$ is not a power of 2;
7. $G \cong E^n_6(q)$ where $\eta = \pm$ and $q$ is odd.
$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

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5. $G \cong PSp_{2n}(q)$, where $n \geq 2$ and $q \equiv \pm 3 \pmod 8$;
6. $G \cong PSL^n_{\eta}(q)$, where $n \geq 3$, $\eta = \pm$, $q$ is odd, and $n$ is not a power of 2;
7. $G \cong E_6^\eta(q)$ where $\eta = \pm$ and $q$ is odd.
H is pronormal in G if H and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Theorem 3 (A. Kondrat’ev, N.M., D. Revin, 2015). All subgroups of odd index are pronormal in the following simple groups:

1. $\text{Alt}(n)$, where $n \geq 5$;
2. sporadic groups;
3. groups of Lie type over fields of characteristic 2;
4. $\text{PSL}_{2n}(q)$;
5. $\text{PSU}_{2n}(q)$;
6. $\text{PSp}_{2n}(q)$, where $q \not\equiv \pm 3 \pmod{8}$;
7. $\text{PO}^\varepsilon_n(q)$, where $\varepsilon \in \{+, -, \text{empty symbol}\}$;
8. exceptional groups of Lie type not isomorphic to $E_6(q)$ or $^2E_6(q)$. 
$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

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5. $G \cong PSp_{2n}(q)$, where $n \geq 2$ and $q \equiv \pm 3 \pmod{8}$;
6. $G \cong PSL^n_\eta(q)$, where $n \geq 3$, $\eta = \pm$, $q$ is odd, and $n$ is not a power of 2;
7. $G \cong E_6^\eta(q)$ where $\eta = \pm$ and $q$ is odd.
Classification Problem

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Theorem 4 (A. Kondrat’ev, N.M., D. Revin, 2016). Let $G = PSp_n(q)$, where $q \equiv \pm 3 \pmod{8}$ and $n \not\in \{2^m, 2^m(2^{2k} + 1) \mid m, k \in \mathbb{N}\}$. Then $G$ contains a nonpronormal subgroup of odd index.

Problem. Classify simple groups in which all subgroups of odd index are pronormal.

Theorem 5 (A. Kondrat’ev, N.M., D. Revin, 2017)**. Let $G$ be a nonabelian simple group, $S \in Syl_2(G)$, and $C_G(S) \leq S$. Then exactly one of the following statements holds:

1. The subgroups of odd index are pronormal in $G$;
2. $G \cong PSp_{2n}(q)$, where $q \equiv \pm 3 \pmod{8}$ and $n$ is not of the form $2^w$ or $2^w(2^{2k} + 1)$.

**Proof was based on joint results by W. Guo, N.M., and D. Revin.
Classification Problem

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Sketch of Proof

$G = PSp_n(q)$, where $q \equiv \pm 3 \pmod{8}$ and $n \in \{2^m, 2^m(2^{2k} + 1) \mid m, k \in \mathbb{N}\}$;

$H \leq G$ and $|G : H|$ is odd;

$S \in Syl_2(G)$ such that $S \leq H$;

$g \in N_G(S)$ and $K = \langle H, H^g \rangle$;

$K = G \Rightarrow H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$;

$K \neq G \Rightarrow \exists M: K \leq M$ and $M$ is maximal in $G$;

Do we know $M$?
Classification of Maximal Subgroups of Odd Index in Finite Simple Groups

M. Liebeck and J. Saxl (1985) and W. Kantor (1987)

Let \( m = \sum_{i=0}^{\infty} a_i \cdot 2^i \) and \( n = \sum_{i=0}^{\infty} b_i \cdot 2^i \), where \( a_i, b_i \in \{0, 1\} \).

We write \( m \preceq n \) if \( a_i \leq b_i \) for every \( i \) and \( m \prec n \) if, in addition, \( m \neq n \).

Theorem (N.M., 2008). Maximal subgroups of odd index in \( Sp_{2n}(q) = Sp(V) \), where \( n > 1 \) and \( q \) is odd are the following:

(1) \( Sp_{2n}(q_0) \), where \( q = q_0^r \) and \( r \) is an odd prime;
(2) \( Sp_{2m}(q) \times Sp_{2(n-m)}(q) \), where \( m \prec n \);
(3) \( Sp_{2m}(q) \wr Sym(t) \), where \( n = mt \) and \( m = 2^k \);
(4) \( 2^{1+4^{4}}.Alt(5) \), where \( n = 2 \) and \( q \equiv \pm 3 \pmod{8} \) is a prime.
H is pronormal in G if H and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

Let $X_2$ be the class of all simple groups with self-normalized Sylow 2-subgroups,
$Y_2$ be the class of all groups in which the subgroups of odd index are pronormal.

Let $G$ and $K$ be groups, $H \leq G$ and $A \leq G$. Then

(1) $G \in Y_2 \Rightarrow G/A \in Y_2$
(2) $G \in Y_2 \nRightarrow H \in Y_2$
(3) $G \in Y_2 \nRightarrow A \in Y_2$
(4) $G, K \in Y_2 \nRightarrow G \times K \in Y_2$

even for simple groups!
$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

$\mathbb{X}_2$ is the class of all simple groups with self-normalized Sylow 2-subgroups,

$\mathbb{Y}_2$ is the class of all groups in which the subgroups of odd index are pronormal.

**Theorem 6 (W. Guo, N.M., D. Revin, 2016-2017).** Let $G$ be a group, $A \trianglelefteq G$, $A \in \mathbb{Y}_2$, and $G/A \in \mathbb{X}_2$. Let $T$ be a Sylow 2-subgroup of $A$. Then the following conditions are equivalent:

1. $G \in \mathbb{Y}_2$;
2. $N_G(T)/T \in \mathbb{Y}_2$. 
$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$. If $m = \sum_{i=0}^{\infty} a_i \cdot 2^i$ and $n = \sum_{i=0}^{\infty} b_i \cdot 2^i$, where $a_i, b_i \in \{0, 1\}$.

We write $m \preceq n$ if $a_i \leq b_i$ for every $i$ and $m \prec n$ if, in addition, $m \neq n$.

**Theorem 7 (W. Guo, N.M., D. Revin, 2016-2017).** Let $A$ be an abelian group and $G = \prod_{i=1}^{t} (A \wr Sym(n_i))$, where all the wreath products are natural permutation. Then all subgroups of odd index are pronormal in $G$ if and only if for any positive integer $m$, if $m \prec n_i$ for some $i$, then $\text{h.c.f.}(|A|, m)$ is a power of 2.

**Theorem 8 (W. Guo, N.M., D. Revin, 2016-2017).** Let $G = \prod_{i=1}^{t} PSp_{n_i}(q_i)$, where $n_i = 2^{w_i}$ and $q_i$ is odd for each $i$. Then the subgroups of odd index are pronormal in $G$. 
Classification Problem

\( H \) is pronormal in \( G \) if \( H \) and \( H^g \) are conjugate in \( \langle H, H^g \rangle \) for every \( g \in G \).

**Problem.** Classify all the nonabelian simple groups \( G \) such that \( C_G(S) \not\leq S \), where \( S \in Syl_2(G) \), and all the subgroups of index are pronormal in \( G \).

**Theorem 9 (A. Kondrat’ev, N.M., D. Revin, 2017+).**
Let \( G \) be an exceptional group of Lie type \( E_6^\varepsilon(q) \), where \( q \) is odd and \( \varepsilon \in \{+,-\} \). Then every subgroup of odd index is pronormal in \( G \) if and only if \( 9 \) does not divide \( q - \varepsilon 1 \).

**Theorem 10 (A. Kondrat’ev, N.M., D. Revin, 2017+).**
Let \( G = PSU_n(q) = L^-_n(q) \), where \( q \) is odd. All subgroups of odd index are pronormal in \( G \) if and only if for any positive integer \( m \), if \( m \prec n \), then \( h.c.f.(m, (q + 1)) \) is a power of \( 2 \).

**Conjecture.**
Let \( G = PSL_n(q) = L^+_n(q) \), where \( q \) is odd. All subgroups of odd index are pronormal in \( G \) if and only if for any positive integer \( m \), if \( m \prec n \), then \( h.c.f.(m, q(q - 1)) \) is a power of \( 2 \).
Problems

$H$ is pronormal in $G$ if $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

**Problem A.** Complete the classification of simple groups in which the subgroup of odd index are pronormal.

**Problem B.** Describe direct products of simple groups in which the subgroup of odd index are pronormal.

**Problem C.** Classify non-pronormal subgroup of odd index in simple groups.

**Problem D.** Classify non-pronormal subgroup of odd index in direct products of simple groups.
Thank you for your attention!