The dungeon

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The diagram on the last page shows a finite-state deterministic automaton. This is a machine with a finite set of states, and a finite set of transitions, each transition being a map from the set of states to itself. The machine starts in an arbitrary state, and reads a word over an alphabet consisting of labels for the transitions (Red and Blue in the example); each time it reads a letter, it undergoes the corresponding transition.

A reset word is a word with the property that, if the automaton reads this word, it arrives at the same state, independent of its start state. An automaton which possesses a reset word is called synchronizing.

Not every finite automaton has a reset word. For example, if every transition is a permutation, then every word in the transitions evaluates to a permutation. How do we recognise when an automaton is synchronizing?
Automata

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So an automaton is a transformation semigroup with a distinguished generating set.
Industrial robotics

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Suppose the conveyor has a square tray in which the pieces can lie in any orientation. Simple gadgets can be devised so that the first gadget rotates the square through $90^\circ$ anticlockwise; the second rotates it only if it detects that the projection is pointing towards the top. The set-up can be represented by an automaton with four states and two transitions, see next slide.
Now it can be verified that \textit{BRRRBRRRB} is a reset word (and indeed that it is the shortest possible reset word for this automaton).
The Černý conjecture

This is a special case of the Černý conjecture, made about fifty years ago and still open:

If an $n$-state automaton is synchronizing, then it has a reset word of length at most $(n - 1)^2$.

The above example and the obvious generalisation show that the conjecture, if true, is best possible. The Černý conjecture has been proved in some cases, but the best general upper bound known is $O(n^3)$, due to Pin. Here is a proof of an $O(n^3)$ bound, which does not get the best constant, but illustrates a simple but important principle.

Proposition

An automaton is synchronizing if and only if, for any two states $a, b$, there is a word in the transitions which takes the automaton to the same place starting from either $a$ or $b$. 
A bound

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We only have to check whether there is a path from each vertex on the right (a pair of states) to a vertex on the left (a single state). Such a path (if it exists) has length $O(n^2)$, and we only require $n - 1$ “collapses” of pairs to synchronize.
Graph endomorphisms

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An endomorphism of $\Gamma$ is a homomorphism from $\Gamma$ to itself.

Proposition

- A homomorphism from $K_m$ to $\Gamma$ is an embedding of $K_m$ into $\Gamma$; such a homomorphism exists if and only if $\omega(\Gamma) \geq m$.
- A homomorphism from $\Gamma$ to $K_m$ is a proper colouring of $\Gamma$ with $m$ colours; such a homomorphism exists if and only if $\chi(\Gamma) \leq m$.
- There are homomorphisms in both directions between $\Gamma$ and $K_m$ if and only if $\omega(\Gamma) = \chi(\Gamma) = m$. 
The obstruction to synchronization

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**Theorem**

*Let \( S \) be a transformation monoid on \( \Omega \). Then \( S \) fails to be synchronizing if and only if there exists a non-null graph \( \Gamma \) on the vertex set \( \Omega \) for which \( S \leq \text{End}(\Gamma) \). Moreover, we may assume that \( \omega(\Gamma) = \chi(\Gamma) \).*
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**Proof.**

Given a transformation monoid $S$, we define a graph $\text{Gr}(S)$ in which $x$ and $y$ are joined if and only if there is no element $s \in S$ with $xs = ys$. Show that $S \leq \text{End}(\text{Gr}(S))$, that $\text{Gr}(S)$ has equal clique and chromatic number, and that $S$ is synchronizing if and only if $\text{Gr}(S)$ is null.
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We abuse language by making the following definition. A permutation group $G$ on $\Omega$ is *synchronizing* if, for any map $f$ on $\Omega$ which is not a permutation, the monoid $\langle G, f \rangle$ generated by $G$ and $f$ is synchronizing.
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A permutation group \(G\) on \(\Omega\) is non-synchronizing if and only if there exists a \(G\)-invariant graph \(\Gamma\), not complete or null, which has clique number equal to chromatic number.
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**Theorem**

A permutation group $G$ on $\Omega$ is non-synchronizing if and only if there exists a $G$-invariant graph $\Gamma$, not complete or null, which has clique number equal to chromatic number.

The definition of synchronizing fits our paradigm for permutation group properties: $G$ is synchronizing if and only if it preserves no non-trivial graph with equal clique and chromatic numbers.
Theorem
Let $G$ be a permutation group of degree $n > 2$.

- If $G$ is synchronizing, then it is transitive, primitive, and basic.
- If $G$ is 2-homogeneous, then it is synchronizing.

Proof.
If $G$ fails to be transitive, primitive or basic, then it preserves a non-trivial graph with clique number equal to chromatic number (a Hamming graph in the non-basic case, see below). If $G$ is 2-homogeneous it preserves no non-trivial graphs. □
An example

Let $G$ be the group of degree $n = \binom{m}{2}$ induced by $S_m$ acting on the 2-subsets of $\{1, \ldots, m\}$. Then $G$ is primitive and basic, and not 2-homogeneous, for $m > 4$. 
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There are two non-trivial $G$-invariant graphs: the graph where two pairs are joined if they intersect (aka the triangular graph $T(m)$, or the line graph of $K_m$) and the graph where two pairs are joined if they are disjoint (the Kneser graph $K(m, 2)$). These are the two graphs in the triangular association scheme.
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- $T(n)$ has clique number $m - 1$, a maximum clique consisting of all the pairs containing a fixed point. Its chromatic number is $m - 1$ if $m$ is even, and $m$ if $m$ is odd.
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**Theorem**

For $m \geq 5$, $S_m$ acting on 2-sets is synchronizing if and only if $m$ is odd.
Cores and pseudocores

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A graph \( \Gamma \) is strongly regular if there are numbers \( k, \lambda, \mu \) such that the number of common neighbours of vertices \( v, w \) is \( k; \lambda \) or \( \mu \) according as \( v \) and \( w \) are equal, adjacent, or non-adjacent.

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Suppose that the vertex-transitive graph $\Gamma$ has clique number and chromatic number $m$. Then every proper $m$-colouring of $\Gamma$ has all colour classes of the same size. (The proof is an exercise.)
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Theorem

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Not every primitive group is almost synchronizing. The smallest example is on 45 points, and was discovered by Araújo, Bentz, Cameron, Royle and Schaefer.
Non-synchronizing ranks

An integer $m$ is a non-synchronizing rank for the permutation group $G$ on $\Omega$ if there is a map $f$ of rank $m$ such that $\langle G, f \rangle$ is non-synchronizing.
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It is not difficult to show that a transitive imprimitive permutation group of degree $n$ has at least $(\frac{3}{4} - o(1))n$ non-synchronizing ranks.

Conjecture: A primitive permutation group of degree $n$ has only $o(n)$ non-synchronizing ranks.

The greatest known number of non-synchronizing ranks for a primitive group of degree $n$ is about $\sqrt{n}$, see the ABCRS paper.

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A result about transitive groups

Proposition

Let $G$ be a transitive permutation group on $\Omega$. Suppose that $A$ and $B$ are subsets of $\Omega$ with $|A| \cdot |B| = |\Omega|$. Then the average value of $|A g \cap B|$, over $g \in G$, is 1. In particular, either this intersection is always 1, or there exists $g \in G$ with $A g \cap B = \emptyset$. 

Proof. Hint: Count triples $(a, b, g)$ with $a \in A$, $b \in B$, $g \in G$, with $ag = b$. 

Corollary

If $\Gamma$ is a vertex-transitive graph on $n$ vertices, then $\omega(\Gamma) \cdot \alpha(\Gamma) \leq n$. Here $\alpha(\Gamma)$ is the independence number of $\Gamma$, the size of the largest null subgraph. (A complete subgraph and a null subgraph meet in at most one vertex.)
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Separating groups

A transitive permutation group $G$ on a set $\Omega$ is separating if, given any two subsets $A$ and $B$ of $\Omega$ with $|A| \cdot |B| = |\Omega|$ and $|A|, |B| > 1$, there exists $g \in G$ such that $Ag \cap B = \emptyset$: in other words, $A$ and $B$ can be “separated” by an element of $G$. 
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A separating group is synchronizing.

For, if $G$ is non-synchronizing, let $f$ be a map not synchronized by $G$, with minimal rank; let $A$ be a part of $\text{Ker}(f)$, and $B = \text{Im}(f)$. Then $|A| \cdot |B| = |\Omega|$ and $|Ag \cap B| = 1$ for all $g \in G$. 

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For, if $G$ is non-synchronizing, let $f$ be a map not synchronized by $G$, with minimal rank; let $A$ be a part of $\text{Ker}(f)$, and $B = \text{Im}(f)$. Then $|A| \cdot |B| = |\Omega|$ and $|Ag \cap B| = 1$ for all $g \in G$.

**Theorem**

The transitive group $G$ on $\Omega$ is non-separating if and only if there exists a $G$-invariant graph $\Gamma$ on $\Omega$, not complete or null, such that

$$\omega(\Gamma) \cdot \alpha(\Gamma) = |\Omega|.$$
Separation in the hierarchy

We see that a separating group is synchronizing: for if $G$ is not synchronizing, and every image of $A$ is a transversal for $P$, then taking $B$ to be a part of $P$ we see that separation fails. Furthermore, since non-separation requires a non-trivial $G$-invariant graph, a 2-homogeneous group is separating.
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Quadrics

Let $V$ be a 5-dimensional vector space over a finite field $F$ of odd characteristic, and $Q$ a non-singular quadratic form on $V$. There is a choice of basis such that, in coordinates,

$$Q(x_1, \ldots, x_5) = x_1x_2 + x_3x_4 + x_5^2.$$
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The quadric associated with $Q$ is the set of points in the projective space based on $V$ (that is, 1-dimensional subspaces of $V$) on which $Q$ vanishes. The number of points on the quadric is $(q + 1)(q^2 + 1)$. 
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The quadric associated with $Q$ is the set of points in the projective space based on $V$ (that is, 1-dimensional subspaces of $V$) on which $Q$ vanishes. The number of points on the quadric is $(q + 1)(q^2 + 1)$. The associated orthogonal group $O_5(F)$ acts on the quadric; it is transitive on the points, and has just two orbits on pairs of points, corresponding to orthogonality and non-orthogonality with respect to the associated bilinear form.
The orthogonality graph

Let $\Gamma$ be the graph in which two points are joined if they are orthogonal.
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- the clique number of $\Gamma$ is $(q + 1)$, and the cliques of maximal size are totally singular lines on the quadric (the point sets of 2-dimensional subspaces on which the form vanishes identically – the span of the first and third basis vectors is an example);
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- the independence number of $\Gamma$ is $q^2 + 1$, and the independent sets of maximal size are **ovoids** of the quadric, sets of points meeting every line in exactly one point.
Synchronizing but not separating

We see from this that $O_5(q)$ is not separating. Is it synchronizing?
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A colouring of $\Gamma$ with $q + 1$ colours, on the other hand, is a partition of the quadric into $q + 1$ ovoids. Now, for $|F|$ an odd prime, it has been proved that the only ovoids on this quadric are hyperplane sections (quadrics in 3-dimensional projective space). Any two hyperplanes intersect in a plane, and the corresponding quadrics meet in a conic in the plane; so there are no two disjoint ovoids, and *a fortiori* no partitions into ovoids, in this case. So we have a family of groups which are synchronizing but not separating.
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Note how this simple question in synchronization theory leads to the frontiers of knowledge in finite geometry!
Towards the Černý conjecture?

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First note that the conjecture has been proved in the case where none of the transitions of the automaton are permutations; so we may assume that the transitions include both permutations and non-permutations, and it would be enough to deal with the case where there is a single non-permutation, that is, \( S = \langle G, f \rangle \).
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First note that the conjecture has been proved in the case where none of the transitions of the automaton are permutations; so we may assume that the transitions include both permutations and non-permutations, and it would be enough to deal with the case where there is a single non-permutation, that is, \( S = \langle G, f \rangle \).
Now there exist \( x \) and \( y \) such that \( xf = yf \). So we can reduce the rank of an element \( s \in S \) by postmultiplying it by \( gf \), where \( g \) maps two points in the image of \( s \) to \( x \) and \( y \). At most \( n - 1 \) steps of this kind are required.
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This is not possible in general, but there are stronger conditions on the group $G$ which guarantee the first condition, and there are powerful results about the diameters of Cayley graphs for permutation groups.
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And finally . . .

A survey paper “Between primitive and 2-transitive: synchronization and its friends” by J. Araújo, P. J. Cameron, and B. Steinberg, will appear in the next issue of the European Mathematical Society Surveys, due out soon!