Permutation groups

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$$\alpha(g_1g_2) = (\alpha g_1)g_2.$$
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An action of a group \( G \) on \( \Omega \) is a homomorphism from \( G \) to \( \text{Sym}(\Omega) \); its image is a permutation group on \( \Omega \). Whenever we define a property of a permutation group, we use the name for a property of the group action.
An example

Let \( G \) be the group of automorphisms of the cube, acting on the set \( \Omega \) of vertices, edges and faces of the cube: \(|\Omega| = 26\). The action is faithful, so \( G \) is a permutation group.
Let $G$ be the group of automorphisms of the cube, acting on the set $\Omega$ of vertices, edges and faces of the cube: $|\Omega| = 26$. The action is faithful, so $G$ is a permutation group. Automorphism groups of mathematical objects provide a rich supply of permutation groups. These objects can be of almost any kind.
Orbits and transitivity

Let $G$ be a permutation group on $\Omega$. Define a relation $\sim$ on $\Omega$ by the rule

$$\alpha \sim \beta \text{ if and only if there exists } g \in G \text{ such that } \alpha g = \beta.$$
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In the example, there are three orbits: the 8 vertices, the 12 edges, and the 6 faces.
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So we can say:

A permutation group $G$ on $\Omega$ is transitive if and only if there are no non-trivial $G$-invariant subsets.
Transitive actions

Let $G$ act on $\Omega$, and take $\alpha \in \Omega$. The stabiliser of $\alpha$ in $G$ is the set
\[ \{ g \in G : \alpha g = \alpha \} . \]

It is a subgroup of $G$. 

If $H$ is any subgroup of $G$, the (right) coset space of $H$ in $G$ is the set $G : H$ of right cosets $Hx$ of $H$ in $G$.

Now there is a notion of isomorphism of group actions, and the following theorem holds:

Theorem

Any transitive action of $G$ on $\Omega$ is isomorphic to the action of $G$ on the coset space $G : \alpha$, for $\alpha \in \Omega$.

The actions of $G$ on coset spaces $G : H$ and $G : K$ are isomorphic if and only if $H$ and $K$ are conjugate subgroups of $G$. 
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- The actions of $G$ on coset spaces $G : H$ and $G : K$ are isomorphic if and only if $H$ and $K$ are conjugate subgroups of $G$. 
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Regular permutation groups and Cayley’s Theorem

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Theorem

Every group of order $n$ is isomorphic to a subgroup of $S_n$.

In particular we see that asking a group $G$ to be a transitive permutation group is no restriction on the abstract structure of $G$. 
Primitivity

A transitive permutation group $G$ on $\Omega$ is primitive if the only non-trivial $G$-invariant partitions are the trivial ones (the partition with one part and the partition into singletons).
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A transitive permutation group $G$ on $\Omega$ is **primitive** if the only non-trivial $G$-invariant partitions are the trivial ones (the partition with one part and the partition into singletons). This can be said another way. A **block of imprimitivity** is a subset $B$ of $\Omega$ with the property that, for all $g \in G$, either $Bg = B$ or $Bg \cap B = \emptyset$. Then $G$ is primitive if and only if the only blocks of imprimitivity are $\Omega$, singletons, and the empty set.
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Consider our example $G$, in its transitive action on the vertices of the cube. We see that $G$ is imprimitive; indeed it preserves two non-trivial partitions:

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- the partition into pairs of **antipodal** points (opposite ends of long diagonals);
- the partition into the vertex sets of two interlocking tetrahedra.
Let $G$ be a transitive permutation group on $\Omega$, where $|\Omega| > 1$. Then $G$ is primitive if and only if the stabiliser of a point of $\Omega$ is a maximal proper subgroup of $G$. 

Let $G$ be primitive on $\Omega$. Then every non-trivial normal subgroup of $G$ is transitive.

Let $G$ be primitive on $\Omega$. Then $G$ has at most two minimal normal subgroups; if there are two, then they are isomorphic and non-abelian, and each of them acts regularly.

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Basic groups

A **Cartesian structure** on $\Omega$ is an identification of $\Omega$ with $A^d$, where $A$ is some set. We can regard $A$ as an “alphabet”, and $A^d$ as the set of all words of length $d$ over the alphabet $A$. Then $A^d$ is a metric space, with the **Hamming metric** (used in the theory of error-correcting codes): the distance between two words is the number of positions in which they differ.

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Let $G$ be a primitive permutation group on $\Omega$. We say that $G$ is basic if it preserves no non-trivial Cartesian structure on $\Omega$. 

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A Cartesian structure is non-trivial if $|A| > 1$ and $d > 1$. Let $G$ be a primitive permutation group on $\Omega$. We say that $G$ is basic if it preserves no non-trivial Cartesian structure on $\Omega$. Although this concept is only defined for primitive groups, we see that the imprimitive group we met earlier, the symmetry group of the cube acting on the vertices, does preserve a Cartesian structure. The automorphism group of a Cartesian structure over an alphabet of size 2 is necessarily imprimitive – generalise our argument for the cube to see this.
The O’Nan–Scott Theorem

A permutation group $G$ is called
- **affine** if it acts on a vector space $V$ and its elements are products of translations and invertible linear transformations of $V$, so that $G$ contains all the translations;

Theorem

Let $G$ be a finite basic primitive permutation group. Then $G$ is affine, diagonal, or almost simple.
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I won’t define **diagonal** groups; here’s an example. Let $T$ be a finite simple group. Then $T \times T$, acting on $T$ by the rule

$$x(g,h) = g^{-1}xh$$

for all $x, g, h \in G$,

is a diagonal group. (The stabiliser of the identity is the diagonal subgroup $\{(g,g) : g \in G\}$ of $G \times G$.)
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**Theorem**

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Multiple transitivity

If $G$ acts on $\Omega$, then it has induced actions on the set of $t$-element subsets of $\Omega$, or the set of $t$-tuples of distinct elements of $\Omega$, where $t \leq |\Omega|$.
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A 2-homogeneous group is primitive. (Exercise; proof later.)
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A non-identity group is **simple** if its only normal subgroups are itself and the identity subgroup.
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This theorem has revolutionised finite permutation group theory. I will end with one of its consequences.
Theorem (CFSG)

*All finite 2-transitive groups are explicitly known.*
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The only finite 6-transitive groups are the symmetric and alternating groups.
Indeed, there are only two 5-transitive groups which are not symmetric or alternating, the Mathieu groups $M_{12}$ and $M_{24}$; and only two further 4-transitive groups, the Mathieu groups $M_{11}$ and $M_{23}$. 
Transformation semigroups

We recall the definitions.

▶ A semigroup is a set $S$ with a binary operation $\circ$ satisfying the \textit{associative law}:

$$a \circ (b \circ c) = (a \circ b) \circ c$$

for all $a, b, c \in S$. 

▶ A monoid is a semigroup with an identity $1$, an element satisfying $a \circ 1 = 1 \circ a = a$ for all $a \in S$.

▶ A group is a monoid with inverses, that is, for all $a \in S$ there exists $b \in S$ such that $a \circ b = b \circ a = 1$.

From now on we will write the operation as \textit{juxtaposition}, that is, write $ab$ instead of $a \circ b$, and $a^{-1}$ for the inverse of $a$. 
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Mind the gap between semigroups and groups!

To any semigroup we can add an identity to produce a monoid of size one larger. Nothing like this is possible for groups!

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Mind the gap between semigroups and groups!

To any semigroup we can add an identity to produce a monoid of size one larger. Nothing like this is possible for groups!

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Note that the numbers of $n$-element semigroups and $(n+1)$-element monoids are fairly close; this is because we can add an identity to an $n$-element semigroup to form an $(n+1)$-element monoid. But numbers of groups are much smaller; the group axioms are much tighter!
Two analogues of Sym(Ω)

For a set Ω, let $T(\Omega)$ be the set of all the maps from Ω to itself, with the operation of composition. If $|\Omega| = n$, we write $T(\Omega)$ as $T_n$. Note that $T(\Omega)$ is a monoid; it contains Sym(Ω), and $T(\Omega) \setminus \text{Sym}(\Omega)$ is a semigroup. $T(\Omega)$ is the full transformation semigroup on Ω.
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Also let $I(Ω)$ denote the set of all partial bijections on Ω (bijections between subsets of Ω), with composition ‘where possible’: if $f_i$ has domain $A_i$ for $i = 1, 2$, then $f_1f_2$ has domain $(A_1f_1 \cap A_2)f_1^{-1}$ and range $(A_1f_1 \cap A_2)f_2$. Again, if $|Ω| = n$, we write $I_n$. This is the symmetric inverse semigroup.
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The order of $I_n$ is $\sum_{k=0}^{n} \binom{n}{k}^2 k!$; there is no closed form for this expression.
Regularity

An element $a$ of a semigroup $S$ is regular if there exists $x \in S$ such that $axa = a$. The semigroup $S$ is regular if all its elements are regular. Note that a group is regular, since we may choose $x = a^{-1}$. The semigroup $T_n$ is regular (exercise).
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Regularity is equivalent to a condition which appears formally to be stronger:

[Proposition]

If $a \in S$ is regular, then there exists $b \in S$ such that $aba = a$ and $bab = b$.

[Proof]

Choose $x$ such that $axa = a$, and set $b = xax$. Then $aba = axaxa = axa = a$, $bab = xaxaxax = xaxax = xax = b$. 


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An **idempotent** in a semigroup $S$ is an element $e$ such that $e^2 = e$. Note that, if $axa = a$, then $ax$ and $xa$ are idempotents. In a group, there is a unique idempotent, the identity. By contrast, it is possible for a non-trivial semigroup to be generated by its idempotents.
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Since $S$ is finite, the powers of $a$ are not all distinct: suppose that $a^m = a^{m+r}$ for some $m, r > 0$. Then $a^i = a^{i+tr}$ for all $i \geq m$ and $t \geq 1$; choosing $i$ to be a multiple of $r$ which is at least $m$, we see that $a^i = a^{2i}$, so $a^i$ is an idempotent.
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It follows that a finite monoid with a unique idempotent is a group. For the unique idempotent is the identity; and, if $a^i = 1$, then $a$ has an inverse, namely $a^{i-1}$. 
The semigroup $S$ is an **inverse semigroup** if for each $a \in S$ there exists a unique $b \in S$ such that $aba = a$ and $bab = b$. We say that $b$ is the (von Neumann) inverse of $a$. 

**Inverse semigroups**
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The semigroup $S$ is an inverse semigroup if for each $a \in S$ there exists a unique $b \in S$ such that $aba = a$ and $bab = b$. We say that $b$ is the (von Neumann) inverse of $a$. The symmetric inverse semigroup $I(\Omega)$ is an inverse semigroup. In an inverse semigroup, the idempotents commute, and they form a semilattice under the order relation $e \leq f$ if $ef = fe = f$. In $I(\Omega)$, the semilattice of idempotents is isomorphic to the Boolean lattice of all subsets of $\Omega$. 
Analogues of Cayley’s Theorem

Theorem

An n-element semigroup is isomorphic to a sub-semigroup of $T_{n+1}$. 
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A similar but slightly harder theorem holds for inverse semigroups:

Theorem (Vagner–Preston Theorem)
An $n$-element inverse semigroup is isomorphic to a sub-semigroup of $I_n$. 
Basics of transformation semigroups

Any map \( f : \Omega \rightarrow \Omega \) has an image

\[ \text{Im}(f) = \{ xf : x \in \Omega \}, \]

and a kernel, the equivalence relation \( \equiv_f \) defined by

\[ x \equiv_f y \iff xf = yf, \]

or the corresponding partition of \( \Omega \). (We usually refer to the partition when we speak about the kernel of \( f \), which is denoted \( \text{Ker}(f) \).) The rank \( \text{rank}(f) \) of \( f \) is the cardinality of the image, or the number of parts of the kernel.
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Under composition, we clearly have

\[
\text{rank}(f_1f_2) \leq \min\{\text{rank}(f_1), \text{rank}(f_2)\},
\]

and so the set \( S_m = \{f \in S : \text{rank}(f) \leq m\} \) of elements of a transformation semigroup which have rank at most \( m \) is itself a transformation semigroup.
Suppose that $f_1$ and $f_2$ are transformations of rank $r$. The rank of $f_1 f_2$ is at most $r$. Equality holds if and only if Im$(f_1)$ is a transversal for Ker$(f_2)$, in the sense that it contains exactly one point from each part of the partition Ker$(f_2)$. This combinatorial relation between subsets and partitions is crucial for what follows. Here is one simple consequence.
Idempotents in transformation semigroups

Suppose that $f_1$ and $f_2$ are transformations of rank $r$. The rank of $f_1f_2$ is at most $r$. Equality holds if and only if $\text{Im}(f_1)$ is a transversal for $\text{Ker}(f_2)$, in the sense that it contains exactly one point from each part of the partition $\text{Ker}(f_2)$. This combinatorial relation between subsets and partitions is crucial for what follows. Here is one simple consequence.

**Proposition**

Let $f$ be a transformation of $\Omega$, and suppose that $\text{Im}(f)$ is a transversal for $\text{Ker}(f)$. Then some power of $f$ is an idempotent with rank equal to that of $f$.

For the restriction of $f$ to its image is a permutation, and some power of this permutation is the identity.
Permutation groups and transformation semigroups

Let $S$ be a transformation semigroup whose intersection with the symmetric group is a permutation group $G$. How do properties of $G$ influence properties of $S$. In particular, what can we say if $S = \langle G, a \rangle$ for some non-permutation $a$?
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Here is a sample theorem due to Araújo, Mitchell and Schneider.
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Here is a sample theorem due to Araújo, Mitchell and Schneider.

**Theorem**

Let $G$ be a permutation group on $\Omega$, with $|\Omega| = n$. Suppose that, for any map $f$ on $\Omega$ which is not a permutation, the semigroup $\langle G, f \rangle$ is regular. Then either $G$ is the symmetric or alternating group on $\Omega$, or one of the following occurs:

- $n = 5$, $G = C_5$, $C_5 \rtimes C_2$, or $C_5 \rtimes C_4$;
- $n = 6$, $G = PSL(2, 5)$ or $PGL(2, 5)$;
- $n = 7$, $G = AGL(1, 7)$;
- $n = 8$, $G = PGL(2, 7)$;
- $n = 9$, $G = PGL(2, 8)$ or $P\Gamma L(2, 8)$. 