What is this about?
Study of the Perkins semigroup (original motivation) — Algebra
Motivation

- Study of the Perkins semigroup (original motivation) — Algebra
- Scheduling robots on a path, periodically — Computer Science
Motivation

- Study of the Perkins semigroup (original motivation) — Algebra
- Scheduling robots on a path, periodically — Computer Science
- Generalization of circle graphs — Graph Theory
Motivation

- Study of the Perkins semigroup (original motivation) — Algebra
- Scheduling robots on a path, periodically — Computer Science
- Generalization of circle graphs — Graph Theory
- Modification of graphs studied before — Combinatorics on Words
Motivation

- Study of the Perkins semigroup (original motivation) — Algebra
- Scheduling robots on a path, periodically — Computer Science
- Generalization of circle graphs — Graph Theory
- Modification of graphs studied before — Combinatorics on Words
- Beautiful mathematics — Mathematics
Motivation

- Study of the Perkins semigroup (original motivation) — Algebra
- Scheduling robots on a path, periodically — Computer Science
- Generalization of circle graphs — Graph Theory
- Modification of graphs studied before — Combinatorics on Words
- Beautiful mathematics — Mathematics
- Just fun — Human Science


Relations between graph classes

word-representable

subcubic
planar
2-outerplanar
outerplanar

circle \equiv 2\text{-repr.}

3-colorable

2-inductive
partial 2-trees

2-trees

trees

4-colorable & \text{$K_4$-free}

perfect

comparability

4-colorable

bipartite
3-trees

chordal
split

complete \equiv 1\text{-repr.}
Factor

Any consecutive letters in a word generate a factor of the word. All different factors of the word 113 are 1, 3, 11, 13 and 113.
Basic definitions

Factor

Any consecutive letters in a word generate a factor of the word. All different factors of the word 113 are 1, 3, 11, 13 and 113.

Alternating letters in a word

In the word 23125413241362, the letters 2 and 3 alternate because removing all other letters we obtain 2323232 where 2 and 3 come in alternating order.
Basic definitions

**Factor**

Any consecutive letters in a word generate a factor of the word. All different factors of the word 113 are 1, 3, 11, 13 and 113.

**Alternating letters in a word**

In the word 23125413241362, the letters 2 and 3 alternate because removing all other letters we obtain 2323232 where 2 and 3 come in alternating order.

Also, 1 and 3 do not alternate because removing all other letters we obtain 311313 and the factor 11 breaks the alternating order.
Factor

Any consecutive letters in a word generate a factor of the word. All different factors of the word 113 are 1, 3, 11, 13 and 113.

Alternating letters in a word

In the word 23125413241362, the letters 2 and 3 alternate because removing all other letters we obtain 2323232 where 2 and 3 come in alternating order.

Also, 1 and 3 do not alternate because removing all other letters we obtain 311313 and the factor 11 breaks the alternating order.

Note that removing all letters but 5 and 6 we obtain 56 showing that the letters 5 and 6 alternate (by definition).
All graphs considered by us are **simple** (no **loops**, no **multiple edges**).

### Word-representable graph

A graph $G = (V, E)$ is **word-representable** if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$, $x \neq y$, alternate in $w$ if and only if $xy \in E$. ($w$ **must** contain each letter in $V$)
All graphs considered by us are **simple** (no loops, no multiple edges).

### Word-representable graph

A graph $G = (V, E)$ is **word-representable** if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$, $x \neq y$, alternate in $w$ if and only if $xy \in E$. ($w$ must contain each letter in $V$)

### Word-representant

$w$ is a **word-representant**. We say that $w$ represents $G$. 
All graphs considered by us are simple (no loops, no multiple edges).

A graph $G = (V, E)$ is **word-representable** if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$, $x \neq y$, alternate in $w$ if and only if $xy \in E$. ($w$ must contain each letter in $V$)

$w$ is a **word-representant**. We say that $w$ represents $G$.

**Remark**

We deal with **unlabelled graphs**. However, to apply the definition, we need to label graphs. Any labelling of a graph is **equivalent** to any other labelling because letters in $w$ can always be renamed.
Basic definitions

**Word-representable graph**

A graph $G = (V, E)$ is word-representable if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$, $x \neq y$, alternate in $w$ if and only if $xy \in E$. (*$w$ must contain each letter in $V$*)

**Remark**

The class of word-representable graphs is hereditary. That is, removing a vertex $v$ in a word-representable graph $G$ results in a word-representable graph $G'$. Indeed, if $w$ represents $G$ then $w$ with $v$ removed represents $G'$. 
A graph $G = (V, E)$ is **word-representable** if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$, $x \neq y$, alternate in $w$ if and only if $xy \in E$. ($w$ must contain each letter in $V$)

**Example**

The graph is word-representable.

\[ \text{The graph is word-representable.} \]
A graph $G = (V, E)$ is **word-representable** if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$, $x \neq y$, alternate in $w$ if and only if $xy \in E$. ($w$ must contain each letter in $V$)

**Example**

The graph is word-representable.

Indeed, can be represented by 1213423.
Basic definitions

Word-representable graph

A graph $G = (V, E)$ is word-representable if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$, $x \neq y$, alternate in $w$ if and only if $xy \in E$. ($w$ must contain each letter in $V$)

Example: representing complete graphs and empty graphs

The graph $G$ with vertices $\{1, 2, 3, 4\}$ can be represented by $1234$ or $12341234$. Or by any permutation of $\{1, 2, 3, 4\}$. 
A graph $G = (V, E)$ is **word-representable** if there exists a word $w$ over the alphabet $V$ such that letters $x$ and $y$, $x \neq y$, alternate in $w$ if and only if $xy \in E$. ($w$ must contain each letter in $V$)

**Example: representing complete graphs and empty graphs**

- The graph can be represented by 1234 or 12341234.
- Or by any permutation of $\{1, 2, 3, 4\}$.

- The graph can be represented by 12344321 or 11223344.
**Uniform word**

A **uniform word** is defined as follows:

- **k-uniform word** = each letter occurs *k* times
- 243321442311 is a 3-uniform word
- 23154 is a 1-uniform word or permutation
### Uniform word

A **$k$-uniform word** is defined as each letter occurring $k$ times. For example:

- $243321442311$ is a **3-uniform word**.
- $23154$ is a **1-uniform word** or a permutation.

### $k$-representable graph

A graph is **$k$-representable** if there exists a $k$-uniform word representing it.
**Uniform word**

\(k\)-uniform word \(=\) each letter occurs \(k\) times

243321442311 is a 3-uniform word

23154 is a 1-uniform word or permutation

**k-representable graph**

A graph is \(k\)-representable if there exists a \(k\)-uniform word representing it.

**Theorem (SK, Pyatkin; 2008)**

A graph is word-representable \textbf{iff} it is \(k\)-representable for some \(k\).
$k$-representability and graph’s representation number

**Uniform word**

$k$-uniform word = each letter occurs $k$ times

243321442311 is a 3-uniform word

23154 is a 1-uniform word or permutation

**$k$-representable graph**

A graph is $k$-representable if there exists a $k$-uniform word representing it.

**Theorem (SK, Pyatkin; 2008)**

A graph is word-representable iff it is $k$-representable for some $k$.

**Theorem (SK, Pyatkin; 2008)**

$k$-representability implies $(k + 1)$-representability.
Theorem (SK, Pyatkin 2008)

A graph is word-representable iff it is $k$-representable for some $k$.

Proof.

“$\Leftarrow$” Trivial.
Theorem (SK, Pyatkin 2008)

A graph is word-representable iff it is $k$-representable for some $k$.

Proof.

“$\Leftarrow$” Trivial.

“$\Rightarrow$” Proof by example showing how to extend (to the left) a word-representant to a uniform word-representant:
Theorem (SK, Pyatkin 2008)

A graph is word-representable iff it is \( k \)-representable for some \( k \).

Proof.

“\( \Leftarrow \)” Trivial.

“\( \Rightarrow \)” Proof by example showing how to extend (to the left) a word-representant to a uniform word-representant:

- 3412132154 - a word-representant for a graph;
- 534253412132154 - a uniform word-representant for the same graph.
Theorem (SK, Pyatkin 2008)

A graph is word-representable iff it is $k$-representable for some $k$.

Proof.

“$\Leftarrow$” Trivial.

“$\Rightarrow$” Proof by example showing how to extend (to the left) a word-representant to a uniform word-representant:

- 3412132154 - a word-representant for a graph;
- 3412132154 - initial permutation (in blue) of the letters not occurring the maximum number of times;
Theorem (SK, Pyatkin 2008)

A graph is word-representable iff it is $k$-representable for some $k$.

Proof.

“$\Leftarrow$” Trivial.

“$\Rightarrow$” Proof by example showing how to extend (to the left) a word-representant to a uniform word-representant:

- 3412132154 - a word-representant for a graph;
- 3412132154 - initial permutation (in blue) of the letters not occurring the maximum number of times;
- 34253412132154 - another word-representant for the same graph;
Theorem (SK, Pyatkin 2008)

A graph is word-representable iff it is $k$-representable for some $k$.

Proof.

“$\Leftarrow$” Trivial.

“$\Rightarrow$” Proof by example showing how to extend (to the left) a word-representant to a uniform word-representant:

- 3412132154 - a word-representant for a graph;
- 3412132154 - initial permutation (in blue) of the letters not occurring the maximum number of times;
- 34253412132154 - another word-representant for the same graph;
- 34253412132154 - initial permutation (in blue) of the letters not occurring the maximum number of times;
Theorem (SK, Pyatkin 2008)

A graph is word-representable \iff it is \(k\)-representable for some \(k\).

Proof.

“\(\Leftarrow\)” Trivial.

“\(\Rightarrow\)” Proof by example showing how to extend (to the left) a word-representant to a uniform word-representant:

- \[3412132154\] - a word-representant for a graph;
- \[3412132154\] - initial permutation (in blue) of the letters not occurring the maximum number of times;
- \[34253412132154\] - another word-representant for the same graph;
- \[34253412132154\] - initial permutation (in blue) of the letters not occurring the maximum number of times;
- \[534253412132154\] - a uniform word-representant for the same graph.
Graph’s representation number is the least $k$ such that the graph is $k$-representable. By a theorem above, this notion is well-defined for word-representable graphs. For non-word-representable graphs, we let $k = \infty$. 

**Notation**

Let $R(G)$ denote $G$'s representation number. Also, let $R_k = \{ G : R(G) = k \}$. 

**Observation**

$R_1 = \{ G : G \text{ is a complete graph} \}$. 

S. Kitaev (University of Strathclyde)
**Notation**

Let \( \mathcal{R}(G) \) denote \( G \)'s representation number. Also, let \( \mathcal{R}_k = \{ G : \mathcal{R}(G) = k \} \).

**Graph’s representation number**

Graph’s representation number is the **least** \( k \) such that the graph is \( k \)-representable. By a theorem above, this notion is well-defined for word-representable graphs. For non-word-representable graphs, we let \( k = \infty \).
Graph’s representation number

Graph’s representation number is the least $k$ such that the graph is $k$-representable. By a theorem above, this notion is well-defined for word-representable graphs. For non-word-representable graphs, we let $k = \infty$.

Notation

Let $\mathcal{R}(G)$ denote $G$’s representation number. Also, let $\mathcal{R}_k = \{G : \mathcal{R}(G) = k\}$.

Observation

$\mathcal{R}_1 = \{G : G$ is a complete graph$\}$. 
Empty graphs

If $G$ is an empty graph on at least two vertices then $R(G) = 2.$
Empty graphs

If $G$ is an empty graph on at least two vertices then $\mathcal{R}(G) = 2$.

Trees

Trees on at least three vertices belong to $\mathcal{R}_2$. The idea of a simple inductive proof is shown for the tree in “step 7” below.

<table>
<thead>
<tr>
<th>Step</th>
<th>Word Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>$w_1 = 1212$</td>
</tr>
<tr>
<td>Step 2</td>
<td>$w_2 = 123132$</td>
</tr>
<tr>
<td>Step 3</td>
<td>$w_3 = 12341432$</td>
</tr>
<tr>
<td>Step 4</td>
<td>$w_4 = 5152341432$</td>
</tr>
<tr>
<td>Step 5</td>
<td>$w_5 = 656152341432$</td>
</tr>
<tr>
<td>Step 6</td>
<td>$w_6 = 65617572341432$</td>
</tr>
<tr>
<td>Step 7 (the tree itself)</td>
<td>$w_7 = 6561758782341432$</td>
</tr>
</tbody>
</table>
Lemma

If a \textbf{k-uniform word} $w$ represents a graph $G$, then any \textbf{cyclic shift} of $w$ represents $G$. 

Cycle graphs on at least four vertices belong to $R_2$. E.g. see $C_5$:

1. As step 1, remove the edge 15 and represent the resulting tree as 1213243545.
2. Make one letter cyclic shift (moving the last letter): 5121324354.
3. Swap the first two letters to obtain a word-representant for $C_5$: 1521324354.
Lemma

If a \( k \)-uniform word \( w \) represents a graph \( G \), then any cyclic shift of \( w \) represents \( G \).

Cycle graphs

Cycle graphs on at least four vertices belong to \( R_2 \). E.g. see \( C_5 \):

- As step 1, remove the edge 15 and represent the resulting tree as 1213243545.
Lemma

If a \( k \)-uniform word \( w \) represents a graph \( G \), then any cyclic shift of \( w \) represents \( G \).

Cycle graphs

Cycle graphs on at least four vertices belong to \( \mathcal{R}_2 \). E.g. see \( C_5 \):

- As step 1, remove the edge 15 and represent the resulting tree as 1213243545.
- Make one letter cyclic shift (moving the last letter): 5121324354.
**Lemma**

If a $k$-uniform word $w$ represents a graph $G$, then any cyclic shift of $w$ represents $G$.

---

**Cycle graphs**

Cycle graphs on \textbf{at least four} vertices belong to $\mathcal{R}_2$. E.g. see $C_5$:

- As step 1, remove the edge 15 and represent the resulting tree as 1213243545.
- Make one letter cyclic shift (moving the last letter): 5121324354.
- Swap the first two letters to obtain a word-representant for $C_5$: 1521324354.
Circle graphs
Circle graphs

Theorem (Halldórsson, SK, Pyatkin; 2011)

For a graph $G$ different from a complete graph, $R(G) = 2$ iff $G$ is a circle graph.

Reading the letters clockwise:

$1 \ 2 \ 3 \ 3 \ 4 \ 5 \ 2 \ 1 \ 5 \ 4$  $\Rightarrow$  This is a 2-representation of the graph
Characterization of graphs with representation number 2

Circle graphs

Theorem (Halldórsson, SK, Pyatkin; 2011)

For a graph $G$ different from a complete graph, $R(G) = 2$ iff $G$ is a circle graph.

Reading the letters clockwise:

1 2 3 3 4 5 2 1 5 4

This is a 2-representation of the graph.
Graphs with representation number 3

Two non-equivalent 3-representations (by Konovalov and Linton):
1387296(10)7493541283(10)7685(10)194562
134(10)58679(10)273412835(10)6819726495
Two non-equivalent 3-representations (by Konovalov and Linton):
1387296(10)7493541283(10)7685(10)194562
143(10)58679(10)273412835(10)6819726495

Theorem (Halldórsson, SK, Pyatkin; 2010)
Petersen graph is not 2-representable.
Theorem (SK, Pyatkin; 2008)
Every prism is 3-representable.

Theorem (SK; 2013)
None of the prisms is 2-representable.
Theorem (SK, Pyatkin; 2008)

Every prism is 3-representable.

Theorem (SK; 2013)

None of the prisms is 2-representable.
Theorem (SK, Pyatkin; 2008)
Every prism is 3-representable.

Theorem (SK; 2013)
None of the prisms is 2-representable.
Theorem (SK, Pyatkin; 2008) 3-subdivision of any graph is 3-representable. In particular, for every graph $G$ there exists a 3-representable graph $H$ that contains $G$ as a minor.

Remark In fact, any subdivision of a graph is 3-representable as long as at least two new vertices are added on each edge.
Theorem (SK, Pyatkin; 2008)

3-subdivision of any graph is 3-representable. In particular, for every graph $G$ there exists a 3-representable graph $H$ that contains $G$ as a minor.
Subdivisions of graphs

Theorem (SK, Pyatkin; 2008)

3-subdivision of any graph is 3-representable. In particular, for every graph $G$ there exists a 3-representable graph $H$ that contains $G$ as a minor.

Remark

In fact, any subdivision of a graph is 3-representable as long as at least two new vertices are added on each edge.
Questions to ask

- Are all graphs word-representable?
Questions to ask

- Are all graphs word-representable?
- If not, how do we characterize those graphs that are (non-)word-representable?
Questions to ask

- Are all graphs word-representable?
- If not, how do we characterize those graphs that are (non-)word-representable?
- How many word-representable graphs are there?
Questions to ask

- Are all graphs word-representable?
- If not, how do we characterize those graphs that are (non-)word-representable?
- How many word-representable graphs are there?
- What is graph’s representation number for a given graph? Essentially, what is the minimal length of a word-representant?
Questions to ask

- Are all graphs word-representable?
- If not, how do we characterize those graphs that are (non-)word-representable?
- How many word-representable graphs are there?
- What is graph’s representation number for a given graph? Essentially, what is the minimal length of a word-representant?
- How hard is it to decide whether a graph is word-representable or not? (complexity)
Questions to ask

- Are all graphs word-representable?
- If not, how do we characterize those graphs that are (non-)word-representable?
- How many word-representable graphs are there?
- What is graph’s representation number for a given graph? Essentially, what is the minimal length of a word-representant?
- How hard is it to decide whether a graph is word-representable or not? (complexity)
- Which graph operations preserve (non-)word-representability?
Questions to ask

- Are all graphs word-representable?
- If not, how do we characterize those graphs that are (non-)word-representable?
- How many word-representable graphs are there?
- What is graph’s representation number for a given graph? Essentially, what is the minimal length of a word-representant?
- How hard is it to decide whether a graph is word-representable or not? (complexity)
- Which graph operations preserve (non-)word-representability?
- Which graphs are word-representable in your favourite class of graphs?
Comparability graphs

Transitive orientation

An orientation of a graph is **transitive** if presence of edges $u \rightarrow v$ and $v \rightarrow z$ implies presence of the edge $u \rightarrow z$. 
Comparability graphs

Transitive orientation

An orientation of a graph is **transitive** if presence of edges $u \rightarrow v$ and $v \rightarrow z$ implies presence of the edge $u \rightarrow z$.

Comparability graph

A non-oriented graph is a **comparability graph** if it admits a transitive orientation.
Comparability graphs

Transitive orientation

An orientation of a graph is transitive if presence of edges \( u \to v \) and \( v \to z \) implies presence of the edge \( u \to z \).

Comparability graph

A non-oriented graph is a comparability graph if it admits a transitive orientation.

Smallest non-comparability graph

![Diagram of a non-comparability graph](attachment://non-comparability_graph.png)
Comparability graphs

Transitive orientation

An orientation of a graph is transitive if presence of edges $u \rightarrow v$ and $v \rightarrow z$ implies presence of the edge $u \rightarrow z$.

Comparability graph

A non-oriented graph is a comparability graph if it admits a transitive orientation.

Smallest non-comparability graph

![Diagram of smallest non-comparability graph]

S. Kitaev (University of Strathclyde)
Comparability graphs

Transitive orientation

An orientation of a graph is **transitive** if presence of edges $u \rightarrow v$ and $v \rightarrow z$ implies presence of the edge $u \rightarrow z$.

Comparability graph

A non-oriented graph is a **comparability graph** if it admits a transitive orientation.

Smallest non-comparability graph

![Graphs](image-url)
Comparability graphs

Transitive orientation

An orientation of a graph is transitive if presence of edges $u \rightarrow v$ and $v \rightarrow z$ implies presence of the edge $u \rightarrow z$.

Comparability graph

A non-oriented graph is a comparability graph if it admits a transitive orientation.

Smallest non-comparability graph

![Graphs](image-url)
Comparability graphs

Transitive orientation

An orientation of a graph is transitive if presence of edges $u \rightarrow v$ and $v \rightarrow z$ implies presence of the edge $u \rightarrow z$.

Comparability graph

A non-oriented graph is a comparability graph if it admits a transitive orientation.

Smallest non-comparability graph

![Diagrams of smallest non-comparability graphs](image-url)
### Permutationally representable graph

A graph $G = (V, E)$ is **permutationally representable** if it can be represented by a word of the form $p_1 \cdots p_k$ where $p_i$ is a permutation. We say that $G$ is **permutationally $k$-representable**.
A graph $G = (V, E)$ is **permutationally representable** if it can be represented by a word of the form $p_1 \cdots p_k$ where $p_i$ is a permutation. We say that $G$ is **permutationally $k$-representable**.

**Example**

The graph with vertices $\{1, 2, 3, 4\}$ and edges $\{1, 2\}$, $\{3, 4\}$ is permutationally representable by $124314324123$. 
A graph $G = (V, E)$ is **permutationally representable** if it can be represented by a word of the form $p_1 \cdots p_k$ where $p_i$ is a permutation. We say that $G$ is **permutationally $k$-representable**.

**Example**

\begin{align*}
(2 & \ 3) \\
\ 1 & \ 4
\end{align*}

is permutationally representable by $124314324123$.

**Theorem (SK, Seif; 2008)**

*A graph is permutationally representable iff it is a comparability graph.*
The graph $G$ below is obtained from a graph $H$ by adding an all-adjacent vertex (apex):

$$G = \begin{array}{c}
\text{X} \\
\end{array} \begin{array}{c}
\text{H} \\
\end{array}$$
The graph $G$ below is obtained from a graph $H$ by adding an all-adjacent vertex (apex):

$$G = H$$

Theorem (SK, Pyatkin; 2008)

$G$ is word-representable if and only if $H$ is permutationally representable.
Significance of permutational representability

The graph $G$ below is obtained from a graph $H$ by adding an all-adjacent vertex (apex): 

\[ G = H \]

**Theorem (SK, Pyatkin; 2008)**

$G$ is word-representable iff $H$ is permutationally representable.

**Theorem (SK, Pyatkin; 2008)**

$G$ is word-representable $\Rightarrow$ the neighbourhood of each vertex is permutationally representable (is a comparability graph).
Converse to the last theorem is not true

Theorem (Halldórsson, SK, Pyatkin; 2010)

$G$ is word-representable if and only if the neighbourhood of each vertex is permutationally representable (is a comparability graph).

Minimal counterexamples

co-$(T_2)$
### Maximum clique

A **clique** in an undirected graph is a subset of pairwise adjacent vertices. A **maximum clique** is a clique of the **maximum size**.
Maximum clique

A clique in an undirected graph is a subset of pairwise adjacent vertices. A maximum clique is a clique of the maximum size.

Maximum clique problem

Given a graph $G$, the Maximum Clique problem is to find a maximum clique in $G$. 

Remark

The Maximum Clique problem is NP-complete.
Maximum clique

A clique in an undirected graph is a subset of pairwise adjacent vertices. A maximum clique is a clique of the maximum size.

Maximum clique problem

Given a graph $G$, the Maximum Clique problem is to find a maximum clique in $G$.

Remark

The Maximum Clique problem is NP-complete.
Theorem (Halldórsson, SK, Pyatkin; 2011)

*The Maximum Clique problem is polynomially solvable on word-representable graphs.*
Theorem (Halldórsson, SK, Pyatkin; 2011)

The Maximum Clique problem is polynomially solvable on word-representable graphs.

Proof.

- Each neighbourhood of a word-representable graph $G$ is a comparability graph.
Theorem (Halldórsson, SK, Pyatkin; 2011)

The Maximum Clique problem is polynomially solvable on word-representable graphs.

Proof.

- Each neighbourhood of a word-representable graph $G$ is a comparability graph.
- The Maximum Clique problem is known to be solvable on comparability graphs in polynomial time.
Theorem (Halldórsson, SK, Pyatkin; 2011)

*The Maximum Clique problem is polynomially solvable on word-representable graphs.*

Proof.
- Each neighbourhood of a word-representable graph $G$ is a comparability graph.
- The Maximum Clique problem is known to be solvable on comparability graphs in polynomial time.
- Thus the problem is solvable on $G$ in polynomial time, since any maximum clique belongs to the neighbourhood of a vertex including the vertex itself.
Non-word-representable graphs

A general construction via adding an apex

Smallest non-word-representable graph: The wheel graph $W_5$ (to the left) is the smallest non-word-representable graph. It is the only such graph on 6 vertices.

S. Kitaev (University of Strathclyde)
Non-word-representable graphs

A general construction via adding an apex

Smallest non-word-representable graphs

The wheel graph \( W_5 \) (to the left) is the smallest non-word-representable graph. It is the only such graph on 6 vertices.
Odd wheels

Observation

The cycle graphs $C_{2k+1}$ for $k \geq 2$ are non-comparability graphs ⇒ the odd wheels $W_{2k+1}$ for $k \geq 2$ are non-word-representable.
Odd wheels

Observation

The cycle graphs $C_{2k+1}$ for $k \geq 2$ are non-comparability graphs $\Rightarrow$ the odd wheels $W_{2k+1}$ for $k \geq 2$ are non-word-representable.

Observation

The wheel graph $W_5$ is non-word representable $\Rightarrow$ almost all graphs are non-word-representable (since almost all graphs contain $W_5$ as an induced subgraph).
Non-word-representable graphs

The **minimal non-comparability** graph is on 5 vertices, and thus the construction of non-word-representable graphs above gives a graph with a **vertex of degree at least 5**. Collins, SK and Lozin showed non-word-representability of

---

Collins, SK and Lozin showed non-word-representability of
Non-word-representable graphs

Non-word-representable graphs of maximum degree 4

The minimal non-comparability graph is on 5 vertices, and thus the construction of non-word-representable graphs above gives a graph with a vertex of degree at least 5. Collins, SK and Lozin showed non-word-representability of

![Graph diagram](image)

Triangle-free non-word-representable graphs

Adding an apex to a non-empty graph gives a graph containing a triangle. Are there any triangle-free non-word-representable graphs?

Theorem (Halldórsson, SK, Pyatkin; 2011)

There exist triangle-free non-word-representable graphs.
A regular graph is a graph having degree of each vertex the same. It was found out by Herman Chen that the smallest regular non-word-representable graphs are on 8 vertices.
All 25 non-word-representable graphs on 7 vertices

The following picture was created by Herman Chen.

Ozgur Akgun, Ian Gent, Chris Jefferson found the number of non-word-representable graphs on up to 10 nodes: 1, 25, 929, 68545, 4880093 (ca 42% of all connected graphs)
Theorem (Collins, SK, Lozin; 2017)

The number of $n$-vertex word-representable graphs is $2^{\frac{n^2}{3} + o(n^2)}$.

Proof.

**Proof idea:** Apply to the case of word-representable graphs Alekseev-Bollobás-Thomason Theorem related to asymptotic growth of every hereditary class. Details are skipped due time constraints, but they can be found here:

The area of “Patterns in words and permutations” is popular and fast-growing (at the rate 100+ papers per year). The book to the left published in 2011 contains 800+ references and is a comprehensive introduction to the area.
The area of “Patterns in words and permutations” is popular and fast-growing (at the rate 100+ papers per year). The book to the left published in 2011 contains 800+ references and is a comprehensive introduction to the area.

Merging two areas of research

In the context of word-representable graphs, which graphs can be represented if we require that word-representants must avoid a given pattern or a set of patterns.
A trivial example

Describe graphs representable by words avoiding the pattern 21.

Solution: Clearly, any 21-avoiding word is of the form
\[ w = 11 \cdots 122 \cdots 2 \cdots nn \cdots n. \]
A trivial example

Describe graphs representable by words avoiding the pattern 21.

**Solution:** Clearly, any 21-avoiding word is of the form

$$w = 11 \cdots 122 \cdots 2 \cdots nn \cdots n.$$ 

If a letter $x$ occurs at least twice in $w$ then the respective vertex is isolated. The letters occurring exactly once form a clique (are connected to each other). Thus, 21-avoiding words describe graphs formed by a clique and an independent set.
A trivial example

Describe graphs representable by words avoiding the pattern 21.

**Solution:** Clearly, any 21-avoiding word is of the form 

\[ w = 11 \cdots 122 \cdots 2 \cdots nn \cdots n. \]

If a letter \( x \) occurs at least twice in \( w \) then the respective vertex is isolated. The letters occurring exactly once form a clique (are connected to each other). Thus, 21-avoiding words describe graphs formed by a clique and an independent set.

Papers in this direction

Word-representants avoiding patterns

So far, essentially only **patterns of length 3** were studied, two non-equivalent cases of which are **132-avoiding** and **123-avoiding** words.
So far, essentially only **patterns of length 3** were studied, two **non-equivalent** cases of which are **132-avoiding** and **123-avoiding** words.

**Labeling of graphs does matter!**

The **132-avoiding** word 4321234 represents the graph to the left, while **no 132-avoiding** word represents the other graph.
So far, essentially only **patterns of length 3** were studied, two **non-equivalent** cases of which are **132-avoiding** and **123-avoiding** words.

**Labeling of graphs does matter!**

The **132-avoiding** word 4321234 represents the graph to the left, while **no 132-avoiding** word represents the other graph. Indeed, **no two** letters out of 1, 2 and 3 can occur **once** in a word-representant or else the respective vertices would **not** form an **independent set**.

![Graphs](image-url)
So far, essentially only patterns of length 3 were studied, two non-equivalent cases of which are 132-avoiding and 123-avoiding words.

Labeling of graphs does matter!

The 132-avoiding word 4321234 represents the graph to the left, while no 132-avoiding word represents the other graph. Indeed, no two letters out of 1, 2 and 3 can occur once in a word-representant or else the respective vertices would not form an independent set. Say, w.l.o.g. that 1 and 2 occur at least twice. But then we can find 1 and 2 on both sides of an occurrence of the letter 4, and the patten 132 is inevitable.
Word-representants avoiding patterns

- All graphs
- Word-representable graphs
  - Circle graphs
    - 132-rep.
    - 123-rep.
  - Some trees, cycle graphs, complete graphs
- Trees
- Odd wheels $W_5, W_7, \ldots$
- Prisms
- Disjoint union of two complete graphs on 4 vertices and a 7-star
- Disjoint union of complete graphs on more than 3 vertices
Examples of simple, but useful general type results:

**Theorem (Mandelshtam, 2016)**

Let $G$ be a word-representable graph, which can be represented by a word avoiding a pattern $\tau$ of length $k + 1$. Let $x$ be a vertex in $G$ such that its degree $d(x) \geq k$. Then, any word $w$ representing $G$ that avoids $\tau$ must contain no more than $k$ copies of $x$.
Word-representants avoiding patterns

Examples of simple, but useful general type results:

**Theorem (Mandelshtam, 2016)**

Let $G$ be a word-representable graph, which can be represented by a word avoiding a pattern $\tau$ of length $k + 1$. Let $x$ be a vertex in $G$ such that its degree $d(x) \geq k$. Then, any word $w$ representing $G$ that avoids $\tau$ must contain no more than $k$ copies of $x$.

**Proof.**

If there are at least $k + 1$ occurrences of $x$ in $w$, we get a subword $xw_1x \cdots w_kx$ where $k$ neighbours of $x$ in $G$ occur in each $w_i$. But then $w$ contains all patterns of length $k + 1$, in particular, $\tau$. Contradiction.
Examples of simple, but useful general type results:

**Theorem (Mandelshtam, 2016)**

Let $G$ be a word-representable graph, which can be represented by a word avoiding a pattern $\tau$ of length $k + 1$. Let $x$ be a vertex in $G$ such that its degree $d(x) \geq k$. Then, any word $w$ representing $G$ that avoids $\tau$ must contain no more than $k$ copies of $x$.

**Proof.**

If there are at least $k + 1$ occurrences of $x$ in $w$, we get a subword $xw_1x \cdots w_kx$ where $k$ neighbours of $x$ in $G$ occur in each $w_i$. But then $w$ contains all patterns of length $k + 1$, in particular, $\tau$. Contradiction.

**Corollary (Mandelshtam, 2016)**

Let $w$ be a word-representant for a graph which avoids a pattern of length $k + 1$. If some vertex $y$ adjacent to $x$ has degree at least $k$, then $x$ occurs at most $k + 1$ times in $w$. 
Semi-transitive orientations as the main tool in the theory of word-representable graphs discovered so far

Sergey Kitaev

University of Strathclyde

April 21, 2017
A **shortcut** is an oriented graph that

- is **acyclic** (that it, there are no directed cycles);
- has **at least 4 vertices**;
- has **exactly one source** (no edges coming in), **exactly one sink** (no edges coming out), and a **directed path** from the source to the sink that goes through **every** vertex in the graph;
- has an edge connecting the **source** to the **sink**;
- is **not transitive** (that it, there exist vertices $u$, $v$ and $z$ such that $u \to v$ and $v \to z$ are edges, but there is **no** edge $u \to z$).
Example of a shortcut

The part of the graph in red shows **non-transitivity**. There are **two other violations** of transitivity.
The part of the graph in red shows **non-transitivity**. There are **two other violations** of transitivity.

The **blue edge**, from the **source** to the **sink**, justifies the name “**shortcut**” for this type of graphs. Indeed, instead of going through the **longest directed path** from the **source** to the **sink**, we can shortcut it by going directly through the single edge. The **blue edge** is called **shortcutting edge**.
An orientation of a graph is **semi-transitive** if it is

- **acyclic**, and
- **shortcut-free**.

**Remark**

Any transitive orientation is necessary **semi-transitive**. The converse is not true, e.g. the schematic semi-transitively oriented graph below is not transitively oriented:

Thus semi-transitive orientations **generalize** transitive orientations.
Checking if a given acyclic orientation is semi-transitive

- **no arc – no problem!**
- **there is an arc – each path from \(x\) to \(y\) of length at least 3 must induce transitive orientation**
Finding a semi-transitive orientation

- Pick any edge and orient it arbitrarily.

The process can normally be shortened by completing orientation of quadrilaterals as shown on the next slide, which is unique to avoid cycles and shortcuts.
Finding a semi-transitive orientation

- Pick any edge and orient it arbitrarily.
- After that repeat the following procedure: pick an edge connected to an already oriented edge and branch the process by orienting it in one way and the other way assuming such an orientation does not introduce a cycle or a shortcut. E.g. no branching is required for the following situation:

![Diagram](image)
Semi-transitive orientations

Finding a semi-transitive orientation

- Pick **any** edge and orient it **arbitrarily**.
- After that repeat the following procedure: pick an edge connected to an already oriented edge and **branch** the process by orienting it in one way and the other way assuming such an orientation does **not** introduce a **cycle** or a **shortcut**. E.g. **no** branching is required for the following situation:

  ![Diagram](attachment:image.png)

- The process can normally be shorten by e.g. completing orientation of **quadrilaterals** as shown on **next slide**, which is **unique** to **avoid** cycles and shortcuts.
Finding a semi-transitive orientation

The diagonal in the last case may require branching.
Finding a semi-transitive orientation

The diagonal in the last case may require branching.
A key result in the theory of word-representable graphs

Theorem (Halldórsson, Kitaev, Pyatkin; 2015)

A graph $G$ is word-representable iff $G$ admits a semi-transitive orientation.
A key result in the theory of word-representable graphs

Theorem (Halldórsson, Kitaev, Pyatkin; 2015)

A graph $G$ is word-representable iff $G$ admits a semi-transitive orientation.

Proof.

“$\Leftarrow$” Rather complicated and is omitted. An algorithm was created to turn a semi-transitive orientation of a graph into a word-representant.
A key result in the theory of word-representable graphs

Theorem (Halldórsson, Kitaev, Pyatkin; 2015)

A graph \( G \) is word-representable iff \( G \) admits a semi-transitive orientation.

Proof.

“\( \Leftarrow \)” Rather complicated and is omitted. An algorithm was created to turn a semi-transitive orientation of a graph into a word-representant.

“\( \Rightarrow \)” Proof idea: Given a word, say, \( w = 2421341 \), orient the graph represented by \( w \) by letting \( x \rightarrow y \) be an edge if the leftmost \( x \) is to the left of the leftmost \( y \) in \( w \), to obtain a semi-transitive orientation:

\[
\begin{array}{c}
1 \\
\downarrow \\
3 \\
\downarrow \\
\begin{array}{c}
4 \\
\leftarrow \\
2
\end{array}
\end{array}
\]
The shortest length of a word-representant

An upper bound on the length of a word-representant

Any complete graph is 1-representable. The algorithm turning semi-transitive orientations into word-representants gave:

**Theorem (Halldórsson, Kitaev, Pyatkin; 2015)**

Each non-complete word-representable graph $G$ is $2(n - \kappa(G))$-representable, where $\kappa(G)$ is the size of the maximum clique in $G$. 

A corollary to the last theorem is that the recognition problem of word-representability is in NP. Indeed, any word-representant is of length at most $O(n^2)$, and we need $O(n^2)$ passes through such a word to check alternation properties of all pairs of letters.
The shortest length of a word-representant

An upper bound on the length of a word-representant

Any **complete graph** is 1-representable. The algorithm turning semi-transitive orientations into word-representants gave:

**Theorem (Halldórosson, Kitaev, Pyatkin; 2015)**

Each **non-complete** word-representable graph $G$ is $2(n - \kappa(G))$-representable, where $\kappa(G)$ is the size of the *maximum clique* in $G$.

A corollary to the last theorem

The recognition problem of word-representability is in **NP**. Indeed, any word-representant is of length at most $O(n^2)$, and we need $O(n^2)$ passes through such a word to check alternation properties of all pairs of letters.
The shortest length of a word-representant

An upper bound on the length of a word-representant

Any **complete graph** is 1-representable. The algorithm turning semi-transitive orientations into word-representants gave:

**Theorem (Halldórsson, Kitaev, Pyatkin; 2015)**

*Each non-complete word-representable graph* $G$ *is* $2(n - \kappa(G))$-representable, *where* $\kappa(G)$ *is the size of the maximum clique in* $G$.

A corollary to the last theorem

The **recognition problem** of word-representability is in **NP**. Indeed, any word-representant is of length at most $O(n^2)$, and we need $O(n^2)$ passes through such a word to check alternation properties of **all** pairs of letters. There is an **alternative proof** of this complexity observation by Halldórsson in terms of **semi-transitive orientations**.
Crown graph (Cocktail party graph)

Crown graph $H_{n,n}$ is obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

$H_{1,1}$

$H_{2,2}$

$H_{3,3}$
Graphs requiring long word-representants

Crown graph (Cocktail party graph)

Crown graph $H_{n,n}$ is obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

$H_{1,1}$  
$H_{2,2}$  
$H_{3,3}$

Word-representability of crown graphs

$H_{n,n}$ is a comparability graph $\Rightarrow$ it is permutationally representable. In fact, $H_{n,n}$ requires $n$ permutations to be represented. Can $H_{n,n}$ be represented in a shorter way if not to require permutational representability? E.g. $H_{3,3}$ is 2-representable, while $H_{4,4}$ is 3-dimensional cube (a prism) and is 3-representable.
Graphs requiring long word-representants

Crown graph (Cocktail party graph)

Crown graph $H_{n,n}$ is obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

$H_{1,1}$

$H_{2,2}$

$H_{3,3}$

Theorem (Glen, Kitaev, Pyatkin; 2016)

If $n \geq 5$ then the representation number of $H_{n,n}$ is $\lceil n/2 \rceil$ (that is, one needs $\lceil n/2 \rceil$ copies of each letter to represent $H_{n,n}$ but not fewer).
Crown graph (Cocktail party graph)

Crown graph $H_{n,n}$ is obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching.

$H_{1,1}$  $H_{2,2}$  $H_{3,3}$

Open problem

Is it true that out of all bipartite graphs, crown graphs require longest word-representants?
The “worst” known word-representable graph

The graph $G_n$ is obtained from a crown graph $H_{n,n}$ by adding an apex (an all-adjacent vertex). The representation number of $G_n$ is $\left\lfloor \frac{n}{2} \right\rfloor$, which is the highest known representation number.

$$G_3 = \begin{array}{c}
1 \quad 2 \quad 3 \\
1' \quad 2' \quad 3'
\end{array}$$
Graphs requiring long word-representants

The “worst” known word-representable graph

The graph $G_n$ is obtained from a crown graph $H_{n,n}$ by adding an apex (an all-adjacent vertex). The representation number of $G_n$ is $\lfloor n/2 \rfloor$, which is the highest known representation number.

$$G_3 = \begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
1' & 2' & 3'
\end{array}
\end{array}$$

Open problem

Are there any graphs whose representation requires more than $\lfloor n/2 \rfloor$ copies of each letter? Recall that any word-representable graph can be represented by $2n$ copies of each letter (a bit fewer depending on the size of the maximum clique).
Theorem (Halldórsson, Kitaev, Pyatkin; 2015)

Any 3-colorable graph is word-representable.

Proof. Coloring a 3-colorable graph in three colors Red, Green, and Blue, and orienting the edges as Red → Green → Blue, we obtain a semi-transitive orientation. Indeed, obviously there are no cycles, and because the longest directed path involves only three vertices, there are no shortcuts.
Theorem (Halldórsson, Kitaev, Pyatkin; 2015)

Any 3-colorable graph is word-representable.

Proof.

Coloring a 3-colorable graph in three colors Red, Green and Blue, and orienting the edges as Red $\rightarrow$ Green $\rightarrow$ Blue, we obtain a semi-transitive orientation. Indeed, obviously there are no cycles, and because the longest directed path involves only three vertices, there are no shortcuts.
Some corollaries to the last theorem

Petersen graph is 3-colorable $\Rightarrow$ it is word-representable.
Petersen graph is 3-colorable ⇒ it is word-representable. Recall that two non-equivalent word-representants were found by Konovalov and Linton:

1387296(10)7493541283(10)7685(10)194562
134(10)58679(10)273412835(10)6819726495
Petersen graph is 3-colorable \( \Rightarrow \) it is word-representable. Recall that two non-equivalent word-representants were found by Konovalov and Linton:

\[
1387296(10)7493541283(10)7685(10)194562
134(10)58679(10)273412835(10)6819726495
\]

**Theorem (Halldórsson, Kitaev, Pyatkin; 2011)**

*Triangle-free planar graphs* are word-representable.

**Proof.**

By Grötzch’s theorem, *every* triangle-free planar graph is 3-colorable.
Some corollaries to the last theorem

Optimization problems

The following optimization problems are \textbf{NP-hard} on 3-colorable graphs \Rightarrow they are \textbf{NP-hard} on word-representable graphs:

- Dominating Set,
- Vertex Coloring,
- Clique Covering, and
- Maximum Independent Set.
Two complexity results

Theorem (Limouzy; 2014)

*It is an NP-complete problem to recognize whether a given graph is word-representable.*

Remark

The proof of Limouzy’s result appears in the book *Words and Graphs* and it is based on the observation that the class of triangle-free word-representable graphs is exactly the class of cover graphs of posets, recognising which is NP-complete.
Two complexity results

Theorem (Limouzy; 2014)

It is an **NP-complete problem** to recognize whether a given graph is word-representable.

Remark

The proof of Limouzy’s result appears in the book “Words and Graphs” and it is based on the observation that the class of **triangle-free word-representable graphs** is exactly the class of **cover graphs of posets**, recognising which is **NP-complete**.
**Theorem (Limouzy; 2014)**

*It is an NP-complete problem to recognize whether a given graph is word-representable.*

**Remark**

The proof of Limouzy’s result appears in the book “*Words and Graphs*” and it is based on the observation that the class of triangle-free word-representable graphs is exactly the class of cover graphs of posets, recognising which is NP-complete.

**Theorem (Halldórsson, Kitaev, Pyatkin; 2011)**

*Deciding whether a given graph is k-representable, for any fixed k, $3 \leq k \leq \lceil n/2 \rceil$, is NP-complete.*
Replacing a vertex $v$ by a module $H$ (clique or any comparability graph); Neighbors of $v$ become neighbors of all vertices in $H$.

[Proof is straightforward via word-representants]
Graph operations preserving word-representability

- Replacing a vertex $v$ by a module $H$ (clique or any comparability graph); Neighbors of $v$ become neighbors of all vertices in $H$. [Proof is straightforward via word-representants]

- Gluing two word-representable graphs in one vertex:

  \[
  G \bullet + \bullet H = G \circlearrowleft H
  \]

  [Proof is straightforward via semi-transitive orientations]
Graph operations preserving word-representability

- Replacing a vertex \( v \) by a **module** \( H \) (**clique** or **any comparability graph**); Neighbors of \( v \) become neighbors of all vertices in \( H \).
  
  [Proof is straightforward via word-representants]

- **Gluing** two word-representable graphs in **one vertex**:

  \[
  G \quad + \quad H \quad = \quad G \quad \cup \quad H
  \]

  [Proof is straightforward via semi-transitive orientations]

- **Joining** two word-representable graphs by an **edge**:

  \[
  G \quad \& \quad H \quad = \quad G \quad \cup \quad H
  \]

  [Proof is straightforward via semi-transitive orientations]
**Cartesian product** of two graphs (shown by Bruce Sagan):
Graph operations preserving word-representability

- **Cartesian product** of two graphs (shown by Bruce Sagan):

```
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{cartesian_product}}
\end{array}
```

- **Rooted product** of graphs:

```
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{rooted_product}}
\end{array}
```

[Proof is straightforward via semi-transitive orientations]
Graph operations **not** preserving word-representability

- **Taking the complement.** The complement to the cycle graph $C_5$ and an isolated vertex is the **non-word-representable** wheel graph $W_5$. 
Graph operations **not** preserving word-representability

- **Taking the complement.** The complement to the cycle graph $C_5$ and an isolated vertex is the **non-word-representable** wheel graph $W_5$.
- **Gluing** two graphs at an edge or a triangle

![Graph operations](image)
Graph operations **not** preserving word-representability

- Taking the line graph operation. *[Example is on next slide.]*

**Theorem (SK, Salimov, Severs, Úlfarsson; 2011)**

*For any wheel graph* $W_n$ and $n \geq 4$, *the line graph* $L(W_n)$ *is not word-representable.*

**Theorem (SK, Salimov, Severs, Úlfarsson; 2011)**

*For any complete graph* $K_n$ and $n \geq 5$, *the line graph* $L(K_n)$ *is not word-representable.*
**Graph operations not preserving word-representability**

- Taking the line graph operation. [Example is on next slide.]

<table>
<thead>
<tr>
<th>Theorem (SK, Salimov, Severs, Úlfarsson; 2011)</th>
</tr>
</thead>
<tbody>
<tr>
<td>For any wheel graph $W_n$ and $n \geq 4$, the line graph $L(W_n)$ is not word-representable.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (SK, Salimov, Severs, Úlfarsson; 2011)</th>
</tr>
</thead>
<tbody>
<tr>
<td>For any complete graph $K_n$ and $n \geq 5$, the line graph $L(K_n)$ is not word-representable.</td>
</tr>
</tbody>
</table>

**Open problem**

Is the line graph of a non-word-representable graph **always** non-word-representable? (This is the case in all known cases.)
Example of taking the line graph operation

The claw graph; a cycle graph; a path graph $K_{1,3} = C_4 = P_4 = \ldots$

Theorem (SK, Salimov, Severs, Ulfarsson; 2011)

If a connected graph $G$ is not a path graph, a cycle graph or the claw graph $K_{1,3}$, then the line graph $L_n(G)$ is not word-representable for $n > 3$. 

S. Kitaev (University of Strathclyde)
Taking the line graph operation

Example of taking the line graph operation

\begin{align*}
C_4 &= P_4
\end{align*}

Theorem (SK, Salimov, Severs, Ulfarsson; 2011)

If a connected graph $G$ is not a path graph, a cycle graph or the claw graph $K_{1,3}$, then the line graph $L_n(G)$ is not word-representable for $n \geq 4$. 

S. Kitaev (University of Strathclyde)

Semi-transitive orientations

April 21, 2017 21 / 37
The claw graph; a cycle graph; a path graph

K_1;3 = C_4 = P_4

Theorem (SK, Salimov, Severs, Ulfarsson; 2011)

If a connected graph \( G \) is not a path graph, a cycle graph or the claw graph \( K_1;3 \), then the line graph \( L_n(G) \) is not word-representable for \( n \geq 4 \).
Taking the line graph operation

Example of taking the line graph operation

The claw graph; a cycle graph; a path graph

\[ K_{1,3} = \quad C_4 = \quad P_4 = \]
Taking the line graph operation

Example of taking the line graph operation

\[ K_{1,3} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \Rightarrow \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \Rightarrow \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]

The claw graph; a cycle graph; a path graph

\[ K_{1,3} = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \quad C_4 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \quad P_4 = \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \]

Theorem (SK, Salimov, Severs, Úlfarsson; 2011)

If a connected graph \( G \) is not a path graph, a cycle graph or the claw graph \( K_{1,3} \), then the line graph \( L^n(G) \) is not word-representable for \( n \geq 4 \).
Not all planar graphs are word-representable (e.g. odd wheel graphs on at least 5 vertices are non-word-representable).
Not all planar graphs are word-representable (e.g. odd wheel graphs on at least 5 vertices are non-word-representable).

However, all triangle-free planar graphs are 3-colorable and thus are word-representable.
Not all planar graphs are word-representable (e.g. odd wheel graphs on at least 5 vertices are non-word-representable).

However, all triangle-free planar graphs are 3-colorable and thus are word-representable.

Open problem
Characterize (non-)word-representable planar graphs.
Not all planar graphs are word-representable (e.g. odd wheel graphs on at least 5 vertices are non-word-representable).

However, all triangle-free planar graphs are 3-colorable and thus are word-representable.

Open problem

Characterize (non-)word-representable planar graphs.

Towards solving the open problem various, triangulations of planar graphs were considered to be discussed next. Key tools here are 3-colorability and semi-transitive orientations.
Convex polyomino triangulation

**Convex** = no “holes” (missing squares) in a column or a row.
**Convex polyomino triangulation**

**Convex** = no “holes” (missing squares) in a column or a row. Need to watch for **odd wheels** as induced subgraphs.

Theorem (Akrobotu, SK, Mas arova; 2015)
A triangulation of a convex polyomino is word-representable iff it is 3-colorable. There are not 3-colorable word-representable non-convex polyomino triangulations.
Convex polyomino triangulation

**Convex** = no “holes” (missing squares) in a column or a row. Need to watch for **odd wheels** as induced subgraphs.

---

**Theorem (Akrobotu, SK, Masárova; 2015)**

A triangulation of a **convex** polyomino is word-representable iff it is 3-colorable. There are not 3-colorable word-representable **non-convex** polyomino triangulations.
A triangulation of a rectangular polyomino with a single domino tile is word-representable if and only if it is 3-colorable.
Theorem (Glen, SK; 2015)

A triangulation of a rectangular polyomino with a single domino tile is word-representable iff it is 3-colorable.
### Near-triangulation

A **near-triangulation** is a planar graph in which each inner bounded face is a **triangle** (where the outer face may possibly not be a triangle).
Near-triangulation

A near-triangulation is a planar graph in which each inner bounded face is a triangle (where the outer face may possibly not be a triangle).

The following theorem is a far-reaching generalization of the results from the last two slides:

Theorem (Glen; 2016)

A $K_4$-free near-triangulation is 3-colorable iff it is word-representable.
Near-triangulation

A near-triangulation is a planar graph in which each inner bounded face is a triangle (where the outer face may possibly not be a triangle).

The following theorem is a far-reaching generalization of the results from the last two slides:

**Theorem (Glen; 2016)**

A $K_4$-free near-triangulation is 3-colorable iff it is word-representable.

**Open problem**

Characterize word-representable near-triangulations (containing $K_4$).
Triangulations of grid-covered cylinder graphs

Grid-covered cylinder graph

- Diagram of a grid-covered cylinder graph.
- Triangulation of the grid-covered cylinder graph.
Triangulations of grid-covered cylinder graphs

Grid-covered cylinder graph

Triangulation of a grid-covered cylinder graph
Theorem (Chen, SK, Sun; 2016)

A triangulation of a grid-covered cylinder graph with more than three sectors is word-representable iff it contains no $W_5$ or $W_7$ as an induced subgraph.
Theorem (Chen, SK, Sun; 2016)

A triangulation of a grid-covered cylinder graph with more than three sectors is word-representable iff it contains no $W_5$ or $W_7$ as an induced subgraph.

Semi-transitive orientation involved in the proof
Theorem (Chen, SK, Sun; 2016)

A triangulation of a grid-covered cylinder graph with three sectors is word-representable iff it contains as an induced subgraph none of
Theorem (Chen, SK, Sun; 2016)

A triangulation of a grid-covered cylinder graph with three sectors is word-representable iff it contains as an induced subgraph none of

Semi-transitive orientation involved in the proof

S. Kitaev (University of Strathclyde)
Subdivisions of triangular grid graphs

The infinite graph $T^\infty$

S. Kitaev (University of Strathclyde)
The infinite graph $T^\infty$

A triangular grid graph is a subgraph of $T^\infty$, which is formed by all edges bounding finitely many cells. Note that a triangular grid graph does not have to be an induced subgraph of $T^\infty$. 
Subdivisions of triangular grid graphs

Subdivision of a cell

Interior and exterior cells

An edge of a triangular grid graph $G$ shared with a cell in $T_1$ that does not belong to $G$ is a boundary edge. A cell in $G$ that is incident to at least one boundary edge is a boundary cell. A non-boundary cell in $G$ is an interior cell.
Subdivisions of triangular grid graphs

Subdivision of a cell

Interior and exterior cells

An edge of a triangular grid graph $G$ shared with a cell in $T^\infty$ that does not belong to $G$ is a boundary edge. A cell in $G$ that is incident to at least one boundary edge is a boundary cell. A non-boundary cell in $G$ is an interior cell.
Subdivisions of triangular grid graphs

Theorem (Chen, SK, Sun; 2016)

A subdivision of a triangular grid graph $G$ is word-representable if and only if it has no subdivided interior cell.

Minimal non-word-representable subdivision of a triangular grid graph
Theorem (Chen, SK, Sun; 2016)

A subdivision of a triangular grid graph $G$ is word-representable iff it has no induced subgraph isomorphic to the graph above, that is, if $G$ has no subdivided interior cell.
Subdivisions of triangular grid graphs

2-dimensional Sierpiński gasket graph $SG(n)$
Subdivisions of triangular grid graphs

2-dimensional Sierpiński gasket graph $SG(n)$

A semi-transitive orientation of $SG(3)$
Software by Marc Glen to study word-representable graphs

Available at
https://personal.cis.strath.ac.uk/sergey.kitaev/word-representable-graphs.html
Software by Marc Glen to study word-representable graphs

Available at
https://personal.cis.strath.ac.uk/sergey.kitaev/word-representable-graphs.html
Open problems

The software should be of great help in tackling the problems below.

- Which graphs in your favourite class of graphs are word-representable?
- Characterize (non-)word-representable planar graphs.
- Characterize word-representable near-triangulations (containing $K_4$).
- Describe graphs representable by words avoiding a pattern $\tau$ of length $\geq 4$.
- Is it true that out of all bipartite graphs, crown graphs require longest word-representants?
- Are there any graphs whose representation requires more than $\lfloor n/2 \rfloor$ copies of each letter?
- Is the line graph of a non-word-representable graph always non-word-representable?
- Characterize word-representable graphs in terms of forbidden subgraphs.
- Translate a known to you problem from graphs to words representing these graphs, and find an efficient algorithm to solve the obtained problem, and thus the original problem.

[The last two problems are of fundamental importance!]

S. Kitaev (University of Strathclyde)
Exercises for this afternoon

1. Represent the following graph using **two** copies of **each** letter:

```
  1 —— 3 —— 4 —— 7 —— 9
     |         |         |
     2 —— 5 —— 8     6
```

2. The graph below contains **lots of shortcuts**. How many can you see?

```
<Diagram with multiple edges between nodes>
```

3. Use a branching process to show that the partial orientation below **cannot** be extended to a **semi-transitive orientation**:

```
<Diagram with partial orientation>
```
Thank you for your attention!