The independence numbers and the chromatic numbers of random subgraphs of Kneser’s graphs and their generalizations

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The Erdős–Ko–Rado Theorem

### Erdős–Ko–Rado, 1961

Let \([n] = \{1, 2, \ldots, n\}\). Assume that \(\mathcal{F} \subset \binom{[n]}{r}\) with \(r \leq n/2\) is such a collection of \(r\)-subsets that any two of them intersect. Then \(|\mathcal{F}| \leq \binom{n-1}{r-1}\).
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Hilton–Milner, 1967

Let \([n] = \{1, 2, \ldots, n\}\). Assume that \(\mathcal{F} \subset \binom{[n]}{r}\) with \(r \leq n/2\) is such a collection of \(r\)-subsets that any two of them intersect and \(\mathcal{F}\) is not a star. Then \(|\mathcal{F}| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1\).
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Other stability results were proposed by Balogh, Bohman, Mubayi et al. using the notion of a random hypergraph.
A graph-theoretic point of view
The \textit{independence number} $\alpha(G)$ of a graph $G$ is the maximum number of pairwise disjoint vertices of $G$. 
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**Kneser’s graph**

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KG_{n,r} = (V, E), \text{ where } V = \binom{[n]}{r},
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E = \{(A, B): A \cap B = \emptyset\}.
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The *chromatic number* $\chi(G)$ of a graph is the smallest number of colors needed to color all the vertices so that no two vertices of the same color are joined by an edge.
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**Lovász, 1978**

If \( r \leq n/2 \), then \( \chi(KG_{n,r}) = n - 2r + 2 \).
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If \( r \geq 2 \) is fixed and \( n \to \infty \), then w.h.p. \( \alpha(KG_{n,r,1/2}) \sim \binom{n-1}{r-1} \).
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**Bollobás, Narayanan, A.M., 2016**

Fix a real number \( \varepsilon > 0 \) and let \( r = r(n) \) be a natural number such that \( 2 \leq r(n) = o(n^{1/3}) \). Let \( p_c(n, r) = ((r + 1) \log n - r \log r)/(\binom{n-1}{r-1}) \). Then as \( n \to \infty \),

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\mathbb{P} \left( \alpha(KG_{n,r,p}) = \binom{n-1}{r-1} \right) \to \begin{cases} 1 & \text{if } p \geq (1 + \varepsilon)p_c(n, r) \\ 0 & \text{if } p \leq (1 - \varepsilon)p_c(n, r). \end{cases}
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Successively improved by Das and Tran.
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For example, if $g(n)$ is any growing function and $r$ is arbitrary in the range between 2 and $\frac{n}{2} - g(n)$, then for any fixed $p$,

$$\chi(KG_{n,r,p}) \sim \chi(KG_{n,r}).$$
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Frankl, Füredi, 1985

For any fixed $r, s$, there exist $c(r, s), d(r, s)$ such that

$$c(r, s)n^{\max\{s, r-s-1\}} \leq \alpha(G(n, r, s)) \leq d(r, s)n^{\max\{s, r-s-1\}}.$$
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**Frankl, Füredi, 1985**

For any fixed $r, s$ such that $r > 2s + 1$,

$$\alpha(G(n, r, s)) = \binom{n-s-1}{r-s-1} = \Theta(n^{r-s-1}).$$
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Let $r, s$ be fixed and $\varepsilon > 0$. There exists a $\delta = \delta(r, s, \varepsilon)$ such that w.h.p.

$$\alpha(G_{1/2}(n, r, s)) \leq (1 + \varepsilon)\alpha(G(n, r, s)) + \delta \binom{n}{s} \log_2 n.$$
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At the same time, $\binom{n}{s} = \Theta(n^s)$. Thus, w.h.p. we have $\alpha(G_{1/2}(n, r, s)) = O(n^s \log_2 n)$. 
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One can easily show using the first moment method that w.h.p.

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By the way, this agrees perfectly with the results concerning $G(n, p) = G_p(n, 1, 0)$.  

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Now let $r > 2s + 1$. Once again, Frankl and Füredi tell us that

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But this time $(\binom{n}{s}) = o\left(\binom{n-s-1}{r-s-1}\right)$, so that we get w.h.p. 

$$\alpha(G_{1/2}(n, r, s)) \leq (1 + o(1))\binom{n-s-1}{r-s-1},$$

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Asymptotic stability!
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Local conclusion

If $r \leq 2s + 1$, then the independence number of the random graph $G_{1/2}(n, r, s)$ behaves like the independence number of the Erdős–Rényi random graph: w.h.p. it increases log times when compared to the initial independence number. Otherwise, it is stable like its analog for Kneser’s graph.
Main result: more stability?
For Kneser’s graphs, we had complete stability. However, for other $r, s$ with $r > 2s + 1$, we got only asymptotic stability. Is it essential or just a technical problem?
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Now, we don’t have such results. Moreover, they are not true! Let’s take \( G(n, 4, 1) \). The Frankl and Wilson linear algebra method gives the bound

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On the other hand, there are two completely different constructions of independent sets with cardinality of order $n^2$. 
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**Construction 1** is just a kind of a star: fix 2 elements of $[n]$ and take all the 4-tuples that contain them. Here we have $\sim \frac{n^2}{2}$ sets.
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Construction 2 is as follows. Divide \([n]\) into consecutive \(\left\lfloor \frac{n}{2} \right\rfloor\) pairs of elements. Then take all the 4-tuples formed by any two such pairs. This way we get \(\sim \frac{n^2}{8}\) sets.
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Let \(r > 3\) be fixed. Then w.h.p.

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\alpha(G_{1/2}(n, r, 1)) = \alpha(G(n, r, 1)).
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It is very important to emphasize here that the exact value of \(\alpha(G(n, r, 1))\) is unknown for all values of \(r\)!
One more graph
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Theorem (Nagy, 1972).

If $n \equiv 0 \pmod{4}$, then $\alpha(G(n, 3, 1)) = n$. If $n \equiv 1 \pmod{4}$, then $\alpha(G(n, 3, 1)) = n - 1$. If $n \equiv 2, 3 \pmod{4}$, then $\alpha(G(n, 3, 1)) = n - 2$. 

A. Raigorodskii (MIPT, YND)
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Theorem (Balogh, Kostochka, A.M., 2012).

If \( n = 2^k \), then \( \chi(G(n, 3, 1)) = (n - 1)(n - 2)/6 \).
Theorem (Nagy, 1972).

If \( n \equiv 0 \pmod{4} \), then \( \alpha(G(n, 3, 1)) = n \). If \( n \equiv 1 \pmod{4} \), then \( \alpha(G(n, 3, 1)) = n - 1 \). If \( n \equiv 2, 3 \pmod{4} \), then \( \alpha(G(n, 3, 1)) = n - 2 \).

Theorem (Balogh, Kostochka, A.M., 2012).

If \( n = 2^k \), then \( \chi(G(n, 3, 1)) = (n - 1)(n - 2)/6 \).

Theorem (Pyaderkin, A.M., 2016).

W.h.p.

\[ \alpha(G_{1/2}(n, 3, 1)) \sim 2n \log_2 n. \]