Partial Latin rectangles graphs and symmetries of partial Latin rectangles

Rebecca J. Stones (Nankai University, China 🇨🇳); with Raúl M. Falcón (University of Seville, Spain 🇪🇸).

February 25, 2017
Monash conference...

My hometown, and where I did my PhD:

http://www.monash.edu/5icc/
This is what a partial Latin rectangle looks like today...

No symbol is duplicated in any row or column.
This is what a partial Latin rectangle looks like today...

No symbol is duplicated in any row or column.

We have $r = 3$ rows.

We have $s = 5$ columns.

We have $n = 3$ symbols.
This is what a partial Latin rectangle looks like today...

No symbol is duplicated in any row or column.

- We have \( r = 3 \) rows.
- We have \( s = 5 \) columns.
- We have \( n = 3 \) symbols.
- We have weight \( m = 7 \). I.e. 7 non-empty cells.
This is what a partial Latin rectangle looks like today...

No symbol is duplicated in any row or column.

- We have $r = 3$ rows.
- We have $s = 5$ columns.
- We have $n = 3$ symbols.
- We have weight $m = 7$. I.e. 7 non-empty cells.

No row is empty. No column is empty. Every symbol $\{1, 2, \ldots, n\}$ is used at least once.
This is what a partial Latin rectangle looks like today...

No symbol is duplicated in any row or column.

- We have $r = 3$ rows.
- We have $s = 5$ columns.
- We have $n = 3$ symbols.
- We have weight $m = 7$. I.e. 7 non-empty cells.

No row is empty. No column is empty. Every symbol $\{1, 2, \ldots, n\}$ is used at least once.

Rows are labeled $\{1, 2, \ldots, r\}$. Columns are labeled $\{1, 2, \ldots, s\}$. 
Some partial Latin rectangles have symmetries...

For this partial Latin rectangle

if we swap the two rows, and swap columns 1 and 3, and swap columns 2 and 4, we generate the partial Latin rectangle we started off with.
Some partial Latin rectangles have symmetries...

For this partial Latin rectangle

If we swap the two rows, and swap columns 1 and 3, and swap columns 2 and 4, we generate the partial Latin rectangle we started off with.

(This why we don’t want empty rows and columns, and unused symbols. E.g. if there were two empty rows, we can swap them to give an uninteresting symmetry.)
Two types of operations...

- We can permute the rows, columns, and symbols.

\[ \{ (1, 1, 1), (1, 2, 2), (2, 1, 3) \} \]

Our second operation is permuting the coordinates of every entry in the entry set, e.g., if we cyclically permute the coordinates of the entries above, we get:

\[ \{ (1, 1, 1), (2, 2, 1), (1, 3, 2) \} \]

There are \( 3! = 6 \) operations of this kind.

A combination of these two types of operations is called an isotopism.
Two types of operations...

We can permute the rows, columns, and symbols. A combination of these three operations is called an isotopism.

\[ \{(1, 1, 1), (1, 2, 2), (2, 1, 3)\} \]

There are \(3! = 6\) operations of this kind.

Our second operation is permuting the coordinates of every entry in the entry set, e.g., if we cyclically permute the coordinates of the entries above, we get:

\[ \{(1, 1, 1), (2, 2, 1), (1, 3, 2)\} \]

There are \(3! = 6\) operations of this kind.

A combination of these two types of operations is called a paratopism.
Two types of operations...

We can permute the rows, columns, and symbols. A combination of these three operations is called an isotopism. There are $r!s!n!$ operations of this kind.
Two types of operations...

- We can permute the rows, columns, and symbols. A combination of these three operations is called an *isotopism*. There are $r!s!n!$ operations of this kind.
- If cell $(i, j)$ contains symbol $k$, then we define the *entry* $(i, j, k)$.
Two types of operations...

- We can permute the rows, columns, and symbols. A combination of these three operations is called an *isotopism*. There are \( r!s!n! \) operations of this kind.

- If cell \((i, j)\) contains symbol \(k\), then we define the *entry* \((i, j, k)\), and the set of all entries is called the *entry set*.
Two types of operations...

- We can permute the rows, columns, and symbols. A combination of these three operations is called an *isotopism*. There are $r! s! n!$ operations of this kind.
- If cell $(i, j)$ contains symbol $k$, then we define the *entry* $(i, j, k)$, and the set of all entries is called the *entry set*.

\[
\begin{align*}
&\begin{array}{c}
1 \\
3
\end{array} \\&\begin{array}{c}
2 \\
\end{array} \\
&\begin{array}{c}
\end{array}
\end{align*}
\]

\[
\left\{(1, 1, 1), (1, 2, 2), (2, 1, 3)\right\}
\]
Two types of operations...

- We can permute the rows, columns, and symbols. A combination of these three operations is called an isotopism. There are $r!s!n!$ operations of this kind.
- If cell $(i, j)$ contains symbol $k$, then we define the entry $(i, j, k)$, and the set of all entries is called the entry set.

```
1 2
3
←→
{(1, 1, 1),
(1, 2, 2),
(2, 1, 3)}
```

Our second operation is permuting the coordinates of every entry in the entry set,
Two types of operations...

- We can permute the rows, columns, and symbols. A combination of these three operations is called an isotopism. There are $r!s!n!$ operations of this kind.

- If cell $(i, j)$ contains symbol $k$, then we define the entry $(i, j, k)$, and the set of all entries is called the entry set.

Our second operation is permuting the coordinates of every entry in the entry set, e.g., if we cyclically permute the coordinates of the entries above, we get:
Two types of operations...

We can permute the rows, columns, and symbols. A combination of these three operations is called an isotopism. There are $r!s!n!$ operations of this kind.

If cell $(i,j)$ contains symbol $k$, then we define the entry $(i,j,k)$, and the set of all entries is called the entry set.

Our second operation is permuting the coordinates of every entry in the entry set, e.g., if we cyclically permute the coordinates of the entries above, we get:

There are $3! = 6$ operations of this kind.
Two types of operations...

We can permute the rows, columns, and symbols. A combination of these three operations is called an *isotopism*. There are $r! s! n!$ operations of this kind.

If cell $(i, j)$ contains symbol $k$, then we define the *entry* $(i, j, k)$, and the set of all entries is called the *entry set*.

\[
\begin{array}{ccc}
1 & 2 & \rightarrow \\
3 & & \\
\end{array}
\leftrightarrow\{(1, 1, 1), (1, 2, 2), (2, 1, 3)\}
\]

Our second operation is permuting the coordinates of every entry in the entry set, e.g., if we cyclically permute the coordinates of the entries above, we get:

\[
\begin{array}{ccc}
1 & & 2 \\
& 1 & \leftarrow \\
\end{array}
\leftrightarrow\{(1, 1, 1), (2, 2, 1), (1, 3, 2)\}
\]

There are $3! = 6$ operations of this kind.

A combination of these two types of operations is called an *paratopism*.
Let $S_t$ denote the symmetric group on $\{1, 2, \ldots, t\}$. 
Let $S_t$ denote the symmetric group on $\{1, 2, \ldots, t\}$. The group

$$(S_r \times S_s \times S_n) \rtimes S_3$$

operates on the set of weight-$m$ partial Latin rectangles $L = (l_{ij})_{r \times s}$. ...
Let $S_t$ denote the symmetric group on $\{1, 2, \ldots, t\}$. The group

$$(S_r \times S_s \times S_n) \rtimes S_3$$

operates on the set of weight-$m$ partial Latin rectangles $L = (l_{ij})_{r \times s}$ ...

... with

$$\theta = (\alpha, \beta, \gamma; \delta)$$

mapping $L$ to the partial Latin rectangle defined by:
Let $S_t$ denote the symmetric group on \{1, 2, \ldots, t\}. The group $(S_r \times S_s \times S_n) \rtimes S_3$ operates on the set of weight-$m$ partial Latin rectangles $L = (l_{ij})_{r \times s} \ldots \quad \text{... with} \quad \theta = (\alpha, \beta, \gamma; \delta)$ mapping $L$ to the partial Latin rectangle defined by:

First, we permute the rows of $L$ according to $\alpha$, the columns according to $\beta$, and the symbols according to $\gamma$, giving the partial Latin square $L' = (l'_{ij})$. 

Formality...
Formality...

Let $S_t$ denote the symmetric group on $\{1, 2, \ldots, t\}$. The group

$$(S_r \times S_s \times S_n) \rtimes S_3$$

operates on the set of weight-$m$ partial Latin rectangles $L = (l_{ij})_{r \times s} \ldots$ ...

... with

$$\theta = (\alpha, \beta, \gamma; \delta)$$

mapping $L$ to the partial Latin rectangle defined by:

- First, we permute the rows of $L$ according to $\alpha$, the columns according to $\beta$, and the symbols according to $\gamma$, giving the partial Latin square $L' = (l'_{ij})$.

- Then, we permute the coordinates of each entry in $L'$ according to $\delta$, i.e., if $(e_1, e_2, e_3)$ is an entry of $L'$, then it maps to $(e_{\delta(1)}, e_{\delta(2)}, e_{\delta(3)})$. 
Technically, this is *not* a group action, as we don’t preserve the dimensions of the partial Latin rectangle.

$$\begin{array}{l}
\begin{array}{llll}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array} \\
\begin{array}{llll}
\text{(id, (13), id; id)} \\
\text{i.e., swap columns 1 and 3} \\
\end{array} \\
\begin{array}{llll}
\begin{array}{llll}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array} \\
\begin{array}{llll}
\text{(id, id, id; (12))} \\
\text{i.e., transpose} \\
\end{array} \\
\begin{array}{llll}
\begin{array}{llll}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array} \\
\end{array}
\end{array}$$

But if we restrict to the operations that preserve the dimensions \((r, s, n)\), we indeed have a group action. And it’s okay to talk about stabilizers under this group action. E.g. above, we get a group action by restricting to paratopisms \((\alpha, \beta, \gamma; \delta)\) with \(\delta \in \{\text{id}, (13)\}\) because:

- \(r = 2\) rows;
- \(n = 2\) symbols.
Technically, this is *not* a group action, as we don’t preserve the dimensions of the partial Latin rectangle.

\[ \begin{array}{cc}
1 & 2 \\
\hline
1 & 2
\end{array} \]

\[ (id,(13),id;id) \]

i.e., swap columns 1 and 3

\[ \begin{array}{cc}
1 & 2 \\
\hline
1 & 2
\end{array} \]

\[ (id,id,id;(12)) \]

i.e., transpose

\[ \begin{array}{cc}
1 & 2 \\
\hline
1 & 2
\end{array} \]
Technically, this is *not* a group action, as we don’t preserve the dimensions of the partial Latin rectangle.

But if we restrict to the operations that preserve the dimensions \((r, s, n)\), we indeed have a group action.
Technically, this is *not* a group action, as we don’t preserve the dimensions of the partial Latin rectangle.

But if we restrict to the operations that preserve the dimensions \((r, s, n)\), we indeed have a group action. And it’s okay to talk about stabilizers under this group action.
Technically, this is *not* a group action, as we don’t preserve the dimensions of the partial Latin rectangle.

But if we restrict to the operations that preserve the dimensions \((r, s, n)\), we indeed have a group action. And it’s okay to talk about stabilizers under this group action.

E.g. above, we get a group action by restricting to paratopisms \((\alpha, \beta, \gamma; \delta)\) with \(\delta \in \{\text{id}, (13)\} \) [because: \(r = 2\) rows; \(n = 2\) symbols].
Two symmetry groups...

Autoparatopism group $\text{apar}(L)$ is the stabilizer subgroup of $(S_r \times S_s \times S_n) \rtimes S_3$...
Two symmetry groups...

- **Autoparatopism group** \( \text{apar}(L) \) is the stabilizer subgroup of \((S_r \times S_s \times S_n) \rtimes S_3\)...

- **Autotopism group** \( \text{atop}(L) \) is the stabilizer subgroup of \((S_r \times S_s \times S_n) \rtimes \langle \text{id} \rangle\)...

**Question**

Given two groups \( H_1 \) and \( H_2 \), does there exist a partial Latin rectangle \( L \) with \( \text{apar}(L) \cong H_1 \) and \( \text{atop}(L) \cong H_2 \)?

There are some obvious "no" instances; e.g. we obviously need \( H_2 \) isomorphic to a subgroup of \( H_1 \). There's some slightly less obvious "no" instances...
Two symmetry groups...

Autoparatopism group $\text{apar}(L)$ is the stabilizer subgroup of $(S_r \times S_s \times S_n) \rtimes S_3$...

Autotopism group $\text{atop}(L)$ is the stabilizer subgroup of $(S_r \times S_s \times S_n) \rtimes \langle \text{id} \rangle$...

Question

Given two groups $H_1$ and $H_2$, does there exist a partial Latin rectangle $L$ with $\text{apar}(L) \cong H_1$ and $\text{atop}(L) \cong H_2$?
Two symmetry groups...

Autoparatopism group $\text{apar}(L)$ is the stabilizer subgroup of $(S_r \times S_s \times S_n) \rtimes S_3$...

Autotopism group $\text{atop}(L)$ is the stabilizer subgroup of $(S_r \times S_s \times S_n) \rtimes \langle \text{id} \rangle$...

Question

Given two groups $H_1$ and $H_2$, does there exist a partial Latin rectangle $L$ with $\text{apar}(L) \cong H_1$ and $\text{atop}(L) \cong H_2$?

There are some obvious “no” instances; e.g. we obviously need $H_2$ isomorphic to a subgroup of $H_1$. 
Two symmetry groups...

**Autoparatopism group** $\text{apar}(L)$ is the stabilizer subgroup of $(S_r \times S_s \times S_n) \rtimes S_3$...

**Autotopism group** $\text{atop}(L)$ is the stabilizer subgroup of $(S_r \times S_s \times S_n) \rtimes \langle \text{id} \rangle$...

**Question**

**Given two groups** $H_1$ and $H_2$, does there exist a partial Latin rectangle $L$ with $\text{apar}(L) \cong H_1$ and $\text{atop}(L) \cong H_2$?

There are some obvious “no” instances; e.g. we obviously need $H_2$ isomorphic to a subgroup of $H_1$. There’s some slightly less obvious “no” instances ...
Important necessary condition

Lemma
\( \text{atop}(L) \) is a normal subgroup of \( \text{apar}(L) \) ...
Important necessary condition

Lemma

\textit{atop}(L) \textit{is a normal subgroup of apar}(L) ...

... \textit{and moreover}

\[ \text{apar}(L)/\text{atop}(L) \cong \{ \delta \in S_3 : \exists (\alpha, \beta, \gamma; \delta) \in \text{apar}(L) \}. \]
Important necessary condition

Lemma

\textit{atop}(L) \textit{is a normal subgroup of} \textit{apar}(L) \textit{...}

\textit{... and moreover}

\[ \textit{apar}(L)/\textit{atop}(L) \cong \{ \delta \in S_3 : \exists (\alpha, \beta, \gamma; \delta) \in \textit{apar}(L) \}. \]

\textit{Hence,} \textit{apar}(L)/\textit{atop}(L) \textit{is isomorphic to a subgroup of} S_3, \textit{i.e., one of} \langle \text{id} \rangle, C_2, C_3 \textit{or} S_3.
Important necessary condition

Lemma
\( \text{atop}(L) \) is a normal subgroup of \( \text{apar}(L) \) ... 

... and moreover

\[
\text{apar}(L)/\text{atop}(L) \cong \{ \delta \in S_3 : \exists (\alpha, \beta, \gamma; \delta) \in \text{apar}(L) \}.
\]

Hence, \( \text{apar}(L)/\text{atop}(L) \) is isomorphic to a subgroup of \( S_3 \), i.e., one of \( \langle \text{id} \rangle, C_2, C_3 \text{ or } S_3 \).

Question

Given a group \( H_1 \) with a normal subgroup \( H_2 \) satisfying \( H_1/H_2 \cong \langle \text{id} \rangle, C_2, C_3 \text{ or } S_3 \), does there exist a partial Latin rectangle \( L \) with \( \text{apar}(L) \cong H_1 \) and \( \text{atop}(L) \cong H_2 \)?
Important necessary condition

Lemma

atop(L) is a normal subgroup of apar(L) ...

... and moreover

\[ \text{apar}(L)/\text{atop}(L) \cong \{ \delta \in S_3 : \exists (\alpha, \beta, \gamma; \delta) \in \text{apar}(L) \}. \]

Hence, apar(L)/atop(L) is isomorphic to a subgroup of S_3, i.e., one of \( \langle \text{id} \rangle, C_2, C_3 \) or \( S_3 \).

Question

Given a group \( H_1 \) with a normal subgroup \( H_2 \) satisfying \( H_1/H_2 \cong \langle \text{id} \rangle, C_2, C_3 \) or \( S_3 \), does there exist a partial Latin rectangle \( L \) with apar(L) \( \cong H_1 \) and atop(L) \( \cong H_2 \)?

i.e., is the above necessary condition sufficient?
Important necessary condition

Lemma

atop(L) is a normal subgroup of apar(L) ...

... and moreover

\[ \text{apar}(L)/\text{atop}(L) \cong \{ \delta \in S_3 : \exists (\alpha, \beta, \gamma; \delta) \in \text{apar}(L) \}. \]

Hence, apar(L)/atop(L) is isomorphic to a subgroup of S_3, i.e., one of \( \langle \text{id} \rangle \), C_2, C_3 or S_3.

Question

Given a group \( H_1 \) with a normal subgroup \( H_2 \) satisfying \( H_1/H_2 \cong \langle \text{id} \rangle, C_2, C_3 \) or \( S_3 \), does there exist a partial Latin rectangle \( L \) with apar\( (L) \cong H_1 \) and atop\( (L) \cong H_2 \)?

i.e., is the above necessary condition sufficient?

The answer is “yes” when \( H_1 = H_2 \) (Phelps 1979, S. 2013).
Any weight-$m$ partial Latin rectangle $L = (l_{ij})$ corresponds to a $m$-vertex partial Latin rectangle graph with:
Partial Latin rectangles graphs

Any weight-$m$ partial Latin rectangle $L = (l_{ij})$ corresponds to a $m$-vertex partial Latin rectangle graph with:

- vertex set equal to the entry set of $L$, and
Partial Latin rectangles graphs

Any weight-$m$ partial Latin rectangle $L = (l_{ij})$ corresponds to a $m$-vertex partial Latin rectangle graph with:

- vertex set equal to the entry set of $L$, and
- an edge between distinct vertices $(i, j, k)$ and $(i', j', k')$ whenever $i = i'$, $j = j'$, or $k = k'$.
Any weight-$m$ partial Latin rectangle $L = (l_{ij})$ corresponds to a $m$-vertex partial Latin rectangle graph with:

- vertex set equal to the entry set of $L$, and
- an edge between distinct vertices $(i, j, k)$ and $(i', j', k')$ whenever $i = i'$, $j = j'$, or $k = k'$.

**Same row:** green edge. **Same column:** orange edge. **Same symbol:** purple edge.
Lemma
An autotopism of a partial Latin rectangle induces an edge-color-preserving automorphism of the corresponding partial Latin rectangle graph.
Lemma
An autotopism of a partial Latin rectangle induces an edge-color-preserving automorphism of the corresponding partial Latin rectangle graph.

Lemma
An autoparatopism of a partial Latin rectangle induces an edge-color-class-preserving automorphism of the corresponding partial Latin rectangle graph.
Lemma
An autotopism of a partial Latin rectangle induces an edge-color-preserving automorphism of the corresponding partial Latin rectangle graph.

Lemma
An autoparatopism of a partial Latin rectangle induces an edge-color-class-preserving automorphism of the corresponding partial Latin rectangle graph.
Lemma
An autotopism of a partial Latin rectangle induces an edge-color-preserving automorphism of the corresponding partial Latin rectangle graph.

Lemma
An autoparatopism of a partial Latin rectangle induces an edge-color-class-preserving automorphism of the corresponding partial Latin rectangle graph.

\[
\begin{align*}
\text{atop} &= \langle (\text{id}, (23), (25); \text{id}) \rangle \quad \text{(size 2)} \\
\text{apar} &= \langle (\text{id}, (23), (25); \text{id}), (\text{id}, (1325), (1523); (23)) \rangle \quad \text{(size 4)} \\
\text{aut} \quad \text{(ignoring edge colors)} \quad \text{(size 8)}
\end{align*}
\]
If $\Gamma(L)$ denotes the partial Latin rectangle graph of $L$, then we have the (partial) subgroup lattices for $\text{apar}(L)$ and $\text{aut}(\Gamma(L))$: 

\[
\begin{align*}
\text{apar}(L) & \subseteq \text{apar}(\Gamma(L)) \\
\text{atop}(L) & \subseteq \text{atop}(\Gamma(L))
\end{align*}
\]
Well, actually...

Annoyingly, a non-trivial autoparatopism might induce the trivial automorphism of the partial Latin rectangle graph.
Annoyingly, a non-trivial autoparatopism might induce the trivial automorphism of the partial Latin rectangle graph. E.g.,

\[
\begin{bmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & 2 & \cdot & \cdot \\
\cdot & \cdot & 3 & \cdot \\
\cdot & \cdot & \cdot & 4
\end{bmatrix}
\]

admits the autoparatopism \((\text{id}, \text{id}, \text{id}; (123))\), which induces the trivial automorphism of the partial Latin rectangle graph.
Annoyingly, a non-trivial autoparatopism might induce the trivial automorphism of the partial Latin rectangle graph. E.g.,

\[
\begin{bmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & 2 & \cdot & \cdot \\
\cdot & \cdot & 3 & \cdot \\
\cdot & \cdot & \cdot & 4
\end{bmatrix}
\]

admits the autoparatopism \((\text{id}, \text{id}, \text{id}; (123))\), which induces the trivial automorphism of the partial Latin rectangle graph.

This doesn’t happen in interesting cases—
Well, actually...

Annoyingly, a non-trivial autoparatopism might induce the trivial automorphism of the partial Latin rectangle graph. E.g.,

\[
\begin{bmatrix}
1 & \cdot & \cdot & \cdot \\
\cdot & 2 & \cdot & \cdot \\
\cdot & \cdot & 3 & \cdot \\
\cdot & \cdot & \cdot & 4
\end{bmatrix}
\]

admits the autoparatopism \((\text{id}, \text{id}, \text{id}; (123))\), which induces the trivial automorphism of the partial Latin rectangle graph.

This doesn’t happen in interesting cases—if we have two entries in some row and two entries in some column, then any non-trivial autoparatopism induces a distinct, non-trivial automorphism.
Trivial autotopism group...

Trivial autotopism group...


We consider the special case of a trivial autotopism group.
Trivial autotopism group...


We consider the special case of a trivial autotopism group. In this case $\text{apar}(L) \cong \langle \text{id} \rangle, C_2, C_3$ or $S_3$. 
Trivial autotopism group...


We consider the special case of a trivial autotopism group. In this case $\text{apar}(L) \cong \langle \text{id} \rangle$, $C_2$, $C_3$ or $S_3$.

**Theorem (example...)**

*Existence:* For $m = 1$ and all $m \geq 3$, there exists a weight-$m$, partial Latin square with a trivial autotopism group, and autoparatopism group isomorphic to $S_3$. 
Trivial autotopism group...


We consider the special case of a trivial autotopism group. In this case $\text{apar}(L) \cong \langle \text{id} \rangle, \ C_2, \ C_3$ or $S_3$.

**Theorem (example...)**

*Existence*: For $m = 1$ and all $m \geq 3$, there exists a weight-$m$, partial Latin square with a trivial autotopism group, and autoparatopism group isomorphic to $S_3$.

🔗 We have similar constructions for $\langle \text{id} \rangle, \ C_2, \text{ and } C_3$. 
Trivial autotopism group...


We consider the special case of a trivial autotopism group. In this case $\text{apar}(L) \cong \langle \text{id} \rangle$, $C_2$, $C_3$ or $S_3$.

**Theorem (example...)**

**Existence:** For $m = 1$ and all $m \geq 3$, there exists a weight-$m$, partial Latin square with a trivial autotopism group, and autoparatopism group isomorphic to $S_3$.

- We have similar constructions for $\langle \text{id} \rangle$, $C_2$, and $C_3$.
- Our constructions are for partial Latin squares, i.e., $r = s = n$. 
Trivial autotopism group...


We consider the special case of a trivial autotopism group. In this case \( \text{apar}(L) \cong \langle \text{id} \rangle, C_2, C_3 \) or \( S_3 \).

**Theorem (example...)**

**Existence:** For \( m = 1 \) and all \( m \geq 3 \), there exists a weight-\( m \), partial Latin square with a trivial autotopism group, and autoparatopism group isomorphic to \( S_3 \).

- We have similar constructions for \( \langle \text{id} \rangle, C_2, \) and \( C_3 \).
- Our constructions are for partial Latin squares, i.e., \( r = s = n \). (For a given \( m \), there exists some \( n \), for which...)
Trivial autotopism group...


We consider the special case of a trivial autotopism group. In this case $\text{apar}(L) \cong \langle \text{id} \rangle$, $C_2$, $C_3$ or $S_3$.

**Theorem (example...)**

*Existence:* For $m = 1$ and all $m \geq 3$, there exists a weight-$m$, partial Latin square with a trivial autotopism group, and autoparatopism group isomorphic to $S_3$.

- We have similar constructions for $\langle \text{id} \rangle$, $C_2$, and $C_3$.
- Our constructions are for partial Latin squares, i.e., $r = s = n$. (For a given $m$, there exists some $n$, for which...)
- Our constructions are connected (i.e., the PLR graph is connected).
Example theorem...

**Theorem**

For \( k \geq 2 \), the partial Latin rectangles

\[
L: \begin{bmatrix}
1 & 2 & \cdots & 4 & 5 \\
3 & 1 & 4 & \cdots & \cdot \\
\end{bmatrix}
\quad \begin{bmatrix}
1 & 2 & 3 & \cdots & 5 & 6 \\
4 & 1 & 2 & 5 & \cdots & \cdot \\
\end{bmatrix}
\quad \begin{bmatrix}
1 & 2 & 3 & 4 & \cdots & 6 & 7 \\
5 & 1 & 2 & 3 & 6 & \cdots & \cdot \\
\end{bmatrix}
\]

\[
M: \begin{bmatrix}
1 & 2 & \cdots & 4 & 5 & \cdots & 7 \\
3 & 1 & 4 & \cdots & 7 & 6 \\
\end{bmatrix}
\quad \begin{bmatrix}
1 & 2 & 3 & \cdots & 5 & 6 & \cdots & 8 \\
4 & 1 & 2 & 5 & \cdots & 8 & 7 \\
\end{bmatrix}
\quad \begin{bmatrix}
1 & 2 & 3 & 4 & \cdots & 6 & 7 & \cdots & 9 \\
5 & 1 & 2 & 3 & 6 & \cdots & 9 & 8 \\
\end{bmatrix}
\]

have trivial autoparatopism groups (and hence trivial autotopism groups).
Proof...

Take the partial Latin rectangle

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & \cdot & 6 & 7 \\
5 & 1 & 2 & 3 & 6 & \cdot & \cdot
\end{bmatrix}
\]
Proof...

Take the partial Latin rectangle

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & \cdot & 6 & 7 \\
5 & 1 & 2 & 3 & 6 & \cdot & \cdot
\end{bmatrix}
\]

Find its partial Latin rectangle graph

![Graph diagram](image-url)
Proof...

Take the partial Latin rectangle

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & \cdot & 6 & 7 \\
5 & 1 & 2 & 3 & 6 & \cdot & \cdot
\end{bmatrix}
\]

Find its partial Latin rectangle graph

Observe it has a trivial automorphism group (ignoring edge colors).
Proof...

Take the partial Latin rectangle

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & \cdot & 6 & 7 \\
5 & 1 & 2 & 3 & 6 & \cdot & \cdot \\
\end{bmatrix}
\]

Find its partial Latin rectangle graph

Observe it has a trivial automorphism group (ignoring edge colors). Therefore the partial Latin rectangle has a trivial autoparatopism group.
Proof...

Take the partial Latin rectangle

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & \cdot & 6 & 7 \\
5 & 1 & 2 & 3 & 6 & \cdot & \cdot
\end{bmatrix}
\]

Find its partial Latin rectangle graph

Observe it has a trivial automorphism group (ignoring edge colors). Therefore the partial Latin rectangle has a trivial autoparatopism group.

This is easy to do for the graph
Proof...

Take the partial Latin rectangle

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & \cdot & 6 & 7 \\
5 & 1 & 2 & 3 & 6 & \cdot & \cdot
\end{bmatrix}
\]

Find its partial Latin rectangle graph

Observe it has a trivial automorphism group (ignoring edge colors). Therefore the partial Latin rectangle has a trivial autoparatopism group.

This is easy to do for the graph—it’s not easy to do for the partial Latin rectangle.
Another technique...

Take the weight-$3k$ partial Latin square $L$ of order $k + 1$ defined by the entry set

$$E(L) = \bigcup_{1 \leq i \leq k} \{(i, i, i + 1), (i, i + 1, i), (i + 1, i, i)\}.$$
Another technique...

Take the weight-3\(k\) partial Latin square \(L\) of order \(k + 1\) defined by the entry set

\[
E(L) = \bigcup_{1 \leq i \leq k} \{(i, i, i + 1), (i, i + 1, i), (i + 1, i, i)\}.
\]

\[
\begin{bmatrix}
2 & 1 \\
1 & \cdot
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & \cdot \\
1 & 3 & 2 \\
\cdot & 2 & \cdot
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot \\
1 & 3 & 2 & \cdot \\
\cdot & 2 & 4 & 3 \\
\cdot & \cdot & 3 & \cdot
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & \cdot & \cdot & \cdot \\
1 & 3 & 2 & \cdot & \cdot \\
\cdot & 2 & 4 & 3 & \cdot \\
\cdot & \cdot & 3 & 5 & 4 \\
\cdot & \cdot & \cdot & 4 & \cdot
\end{bmatrix}
\]

Autoparatopism group contains \((\text{id}, \text{id}, \text{id}; \delta)\) for all \(\delta \in S_3\)
Another technique...

Take the weight-$3k$ partial Latin square $L$ of order $k + 1$ defined by the entry set

$$E(L) = \bigcup_{1 \leq i \leq k} \{(i, i, i + 1), (i, i + 1, i), (i + 1, i, i)\}.$$

\[
\begin{bmatrix}
2 & 1 \\
1 & .
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & . \\
1 & 3 & 2 \\
. & 2 & .
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & . & . \\
1 & 3 & 2 & . \\
. & 2 & 4 & 3 \\
. & . & 3 & .
\end{bmatrix}
\quad
\begin{bmatrix}
2 & 1 & . & . & . \\
1 & 3 & 2 & . & . \\
. & 2 & 4 & 3 & . \\
. & . & 3 & 5 & 4 \\
. & . & . & 4 & .
\end{bmatrix}
\]

Autoparatopism group contains $(\text{id}, \text{id}, \text{id}; \delta)$ for all $\delta \in S_3$. 
The trick we use for the autotopism group...

\[ E(L) = \bigcup_{1 \leq i \leq k} \{(i, i, i + 1), (i, i + 1, i), (i + 1, i, i)\}. \]

**Lemma**

If \( \theta = (\alpha, \beta, \gamma; \text{id}) \) is an autotopism of \( L \) in which \( \alpha, \beta, \) and \( \gamma \) fix 1, then \( \alpha, \beta, \) and \( \gamma \) fix \( \{1, 2, \ldots, k+1\} \) pointwise.
The trick we use for the autotopism group...

\[ E(L) = \bigcup_{1 \leq i \leq k} \{(i, i, i + 1), (i, i + 1, i), (i + 1, i, i)\}. \]

**Lemma**

*If \( \theta = (\alpha, \beta, \gamma; \text{id}) \) is an autotopism of \( L \) in which \( \alpha, \beta, \) and \( \gamma \) fix 1, then \( \alpha, \beta, \) and \( \gamma \) fix \( \{1, 2, \ldots, k + 1\} \) pointwise.*

**Proof.**

*Induction: If \( i \in \{1, 2, \ldots, k\} \) is fixed by \( \alpha, \beta, \) and \( \gamma, \)
The trick we use for the autotopism group...

\[ E(L) = \bigcup_{1 \leq i \leq k} \{(i, i, i + 1), (i, i + 1, i), (i + 1, i, i)\}. \]

**Lemma**

If \( \theta = (\alpha, \beta, \gamma; \text{id}) \) is an autotopism of \( L \) in which \( \alpha, \beta, \) and \( \gamma \) fix 1, then \( \alpha, \beta, \) and \( \gamma \) fix \( \{1, 2, \ldots, k + 1\} \) pointwise.

**Proof.**

*Induction:* If \( i \in \{1, 2, \ldots, k\} \) is fixed by \( \alpha, \beta, \) and \( \gamma, \) then

\[
(i, i, i + 1) \xrightarrow{\theta} (i, i,?),
\]

\[
(i, i + 1, i) \xrightarrow{\theta} (i, ?, i),
\]

\[
(i + 1, i, i) \xrightarrow{\theta} (?, i, i).
\]
The trick we use for the autotopism group...

\[ E(L) = \bigcup_{1 \leq i \leq k} \{(i, i, i + 1), (i, i + 1, i), (i + 1, i, i)\}. \]

Lemma

If \( \theta = (\alpha, \beta, \gamma; \text{id}) \) is an autotopism of \( L \) in which \( \alpha \), \( \beta \), and \( \gamma \) fix 1, then \( \alpha \), \( \beta \), and \( \gamma \) fix \( \{1, 2, \ldots, k + 1\} \) pointwise.

Proof.

Induction: If \( i \in \{1, 2, \ldots, k\} \) is fixed by \( \alpha \), \( \beta \), and \( \gamma \), then

\[
\begin{align*}
(i, i, i + 1) \xrightarrow{\theta} (i, i,?), \\
(i, i + 1, i) \xrightarrow{\theta} (i, ?, i), \\
(i + 1, i, i) \xrightarrow{\theta} (?, i, i).
\end{align*}
\]

But, since \( M \) is a partial Latin square, each “?” must be \( i + 1 \).
The trick we use for the autotopism group...

\[ E(L) = \bigcup_{1 \leq i \leq k} \{(i, i, i + 1), (i, i + 1, i), (i + 1, i, i)\}. \]

**Lemma**

If \( \theta = (\alpha, \beta, \gamma; \text{id}) \) is an autotopism of \( L \) in which \( \alpha, \beta, \) and \( \gamma \) fix 1, then \( \alpha, \beta, \) and \( \gamma \) fix \( \{1, 2, \ldots, k + 1\} \) pointwise.

**Proof.**

*Induction:* If \( i \in \{1, 2, \ldots, k\} \) is fixed by \( \alpha, \beta, \) and \( \gamma \), then

\[
(i, i, i + 1) \overset{\theta}{\mapsto} (i, i,?), \quad (i, i + 1, i) \overset{\theta}{\mapsto} (i, ?, i), \quad (i + 1, i, i) \overset{\theta}{\mapsto} (?, i, i).
\]

But, since \( M \) is a partial Latin square, each “?” must be \( i + 1 \). So \( i + 1 \) is also fixed by \( \alpha, \beta, \) and \( \gamma \). \( \square \)
If we add these entries

\[
\begin{bmatrix}
2 & 1 & \cdots \\
1 & 4 & \\
\cdots & 3 & 2 \\
\cdots & 3 & 4
\end{bmatrix}
\begin{bmatrix}
2 & 1 & \cdots \\
1 & 3 & 2 & \cdots \\
2 & 5 & \cdots & 4 & 3 \\
\cdots & 4 & 5
\end{bmatrix}
\begin{bmatrix}
2 & 1 & \cdots \\
1 & 3 & 2 & \cdots \\
2 & 4 & 3 & \cdots \\
\cdots & 3 & 6 & \cdots & 4 & 3 \\
\cdots & 4 & 5 & \cdots & 6 & 4 \\
\cdots & 5 & 6 & \cdots & 6 & 5 \\
\cdots & 6 & 7
\end{bmatrix}
\]

we get weight \( m \equiv 2 \pmod{3} \) cases.
If we add these entries

\[
\begin{bmatrix}
2 & 1 & \cdot & \cdot \\
1 & 4 & \cdot & \\
\cdot & 3 & 2 & \\
\cdot & 3 & 4 & \\
\end{bmatrix}
\]

we get weight \( m \equiv 2 \pmod{3} \) cases. New autotopisms? Then this graph would have a non-trivial automorphism:

but it doesn’t.
Another editing example...

A construction for the $\text{atop} \cong \langle \text{id} \rangle$ and $\text{apar} \cong C_2$ case:

$k = 1$

\[
\begin{bmatrix}
2 & 1 & \cdots \\
1 & \cdots & \\
\cdot & 3 & 4 \\
\cdots & 4 & 1
\end{bmatrix}
\]

$k = 2$

\[
\begin{bmatrix}
2 & 1 & \cdots \\
1 & 3 & 2 & \cdots \\
\cdot & 2 & \cdots \\
\cdot & 4 & 5 \\
\cdots & 5 & 1
\end{bmatrix}
\]

$k = 3$

\[
\begin{bmatrix}
2 & 1 & \cdots \\
1 & 3 & 2 & \cdots \\
\cdot & 2 & 4 & 3 & \cdots \\
\cdot & 3 & \cdots \\
\cdot & 5 & 6 \\
\cdots & 6 & 1
\end{bmatrix}
\]

$k = 4$

\[
\begin{bmatrix}
2 & 1 & \cdots & \cdots \\
1 & 3 & 2 & \cdots \\
\cdot & 2 & 4 & 3 & \cdots \\
\cdot & 3 & 5 & 4 & \cdots \\
\cdot & 4 & \cdots \\
\cdot & 6 & 7 \\
\cdots & 7 & 1
\end{bmatrix}
\]

This time we add entries that destroy symmetries.
We give constructions of $m$-entry partial Latin rectangles with trivial autotopism groups for all possible autoparatopism groups (up to isomorphism) when:

- $r = s = n$, i.e., partial Latin squares,
- $r = 2$ and $s = n$,
- $r = 2$ and $s \neq n$. 

We expect the partial Latin rectangle graphs material will be useful beyond the scope of this work. E.g. I gave long proof of a trivial autoparatopism group of a partial Latin square in my 2013 European J. Comb. paper, but it's obvious from looking at the graph.

The general question...

Question

For which finite groups, $H_1$ and $H_2$, does there exist a partial Latin rectangle $L \in \text{PLR}(r, s, n; m)$ with $\text{atop}(L) \sim = \langle \text{id} \rangle$ and $\text{apar}(L) \sim = S_3$?

— this time we're taking an "$n$ first approach".
Summary...

We give constructions of $m$-entry partial Latin rectangles with trivial autotopism groups for all possible autoparatopism groups (up to isomorphism) when:

- $r = s = n$, i.e., partial Latin squares,
- $r = 2$ and $s = n$,
- $r = 2$ and $s \neq n$.

We expect the partial Latin rectangle graphs material will be useful beyond the scope of this work.
Summary...

We give constructions of $m$-entry partial Latin rectangles with trivial autotopism groups for all possible autoparatopism groups (up to isomorphism) when:

- $r = s = n$, i.e., partial Latin squares,
- $r = 2$ and $s = n$,
- $r = 2$ and $s \neq n$.

We expect the partial Latin rectangle graphs material will be useful beyond the scope of this work.

E.g. I gave long proof of a trivial autoparatopism group of a partial Latin square in my 2013 European J. Comb. paper, but it’s obvious from looking at the graph.
We give constructions of $m$-entry partial Latin rectangles with trivial autotopism groups for all possible autoparatopism groups (up to isomorphism) when:

- $r = s = n$, i.e., partial Latin squares,
- $r = 2$ and $s = n$,
- $r = 2$ and $s \neq n$.

We expect the partial Latin rectangle graphs material will be useful beyond the scope of this work.

E.g. I gave long proof of a trivial autoparatopism group of a partial Latin square in my 2013 European J. Comb. paper, but it’s obvious from looking at the graph.

The general question...

**Question**

*For which finite groups, $H_1$ and $H_2$, does there exist a partial Latin rectangle $L \in \text{PLR}(r, s, n; m)$ with $\text{atop}(L) \cong H_1$ and $\text{apar}(L) \cong H_2$?*
We give constructions of \( m \)-entry partial Latin rectangles with trivial autotopism groups for all possible autoparatopism groups (up to isomorphism) when:

- \( r = s = n \), i.e., partial Latin squares,
- \( r = 2 \) and \( s = n \),
- \( r = 2 \) and \( s \neq n \).

We expect the partial Latin rectangle graphs material will be useful beyond the scope of this work.

E.g. I gave long proof of a trivial autoparatopism group of a partial Latin square in my 2013 European J. Comb. paper, but it’s obvious from looking at the graph.

The general question...

**Question**

For which finite groups, \( H_1 \) and \( H_2 \), does there exist a partial Latin rectangle \( L \in \text{PLR}(r, s, n; m) \) with \( \text{atop}(L) \cong H_1 \) and \( \text{apar}(L) \cong H_2 \)?

We're currently looking at the \( r = s = n \) case with \( \text{atop}(L) \cong \langle \text{id} \rangle \) and \( \text{apar}(L) \cong S_3 \).
Summary...

We give constructions of \( m \)-entry partial Latin rectangles with trivial autotopism groups for all possible autoparatopism groups (up to isomorphism) when:

- \( r = s = n \), i.e., partial Latin squares,
- \( r = 2 \) and \( s = n \),
- \( r = 2 \) and \( s \neq n \).

We expect the partial Latin rectangle graphs material will be useful beyond the scope of this work.

E.g. I gave long proof of a trivial autoparatopism group of a partial Latin square in my 2013 European J. Comb. paper, but it’s obvious from looking at the graph.

The general question...

**Question**

*For which finite groups, \( H_1 \) and \( H_2 \), does there exist a partial Latin rectangle \( L \in \text{PLR}(r, s, n; m) \) with \( \text{atop}(L) \cong H_1 \) and \( \text{apar}(L) \cong H_2 \)?*

We’re currently looking at the \( r = s = n \) case with \( \text{atop}(L) \cong \langle \text{id} \rangle \) and \( \text{apar}(L) \cong S_3 \). — this time we’re taking an “\( n \) first approach”.