The $\mathbb{Z}_2^n$ Dirac-Dunkl operator and a higher rank Bannai-Ito algebra

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General context

- Leverage the relations between these topics to make advances

-exactly solvable models

-algebraic structures

-special functions

-symmetries
Today’s menu

- Model: The Dirac-Dunkl equation associated to $\mathbb{Z}_2^n$
- Algebraic structures:
  - Lie superalgebra $\mathfrak{osp}(1,2)$
  - The Bannai-Ito algebra at rank $n$
- Symmetries: Nested constants of motion arising as sCasimir operators
- Special functions:
  - Clifford-valued multivariate Jacobi-type orthogonal polynomials stemming from Cauchy-Kovaleskaia extension
  - Multivariate Bannai-Ito polynomials
The $\mathbb{Z}_2^n$ Laplace-Dunkl operator on $\mathbb{R}^n$

- Dunkl operators: families of differential-difference operators associated to finite reflection groups
- Consider reflection group $\mathbb{Z}_2^n = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$
- Dunkl operators $T_1, \ldots, T_n$ (on $\mathbb{R}^n$) defined as
  \[ T_i = \partial_{x_i} + \frac{\mu_i}{x_i} (1 - r_i) \quad i = 1, \ldots, n \]
  with $\mu_1, \ldots, \mu_n > 0$ and
  \[ r_i f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, -x_i, \ldots, x_n) \]
- Clearly $T_i T_j = T_j T_i$ for all $i, j = 1, \ldots, n$
- Laplace-Dunkl operator
  \[ \Delta = \sum_{i=1}^{n} T_i^2 \]
The $\mathbb{Z}_2^n$ Dirac-Dunkl operator on $\mathbb{R}^n$

- Clifford algebra $Cl_n = \langle e_1, \ldots, e_n \rangle$

\[\{e_i, e_j\} = e_ie_j + e_je_i = -2\delta_{ij} \quad i, j = 1, \ldots, n\]

- Dirac-Dunkl operator $D$ and "position" operator $x$

\[D = \sum_{i=1}^{n} e_i T_i \quad x = \sum_{i=1}^{n} e_i x_i\]

- One has

\[D^2 = \Delta \quad x^2 = -||x||^2\]

where $\sum_{i=1}^{n} x_i^2$
Some notation

- Let \([n] = \{1, \ldots, n\}\)
- Let \(A \subset [n]\) and define "intermediate" Dirac-Dunkl and position operators
  \[
  D_A = \sum_{i \in A} e_i T_i \quad x_A = \sum_{i \in A} e_i T_i
  \]
- Similarly
  \[
  \Delta_A = \sum_{i \in A} T_i^2 \quad ||x_A||^2 = \sum_{i \in A} x_i^2
  \]
- Observe that \(D_{[n]} = D\) and \(x_{[n]} = x\)
- Likewise, for \(\ell \leq n\), we shall use
  \[
  D_{[\ell]} = D_{\{1, \ldots, \ell\}} \quad x_{[\ell]} = x_{\{1, \ldots, \ell\}}
  \]
The $\mathfrak{osp}(1,2)$ dynamical symmetry

- Introduce the Euler operator $\mathbb{E}_A = \sum_{i \in A} x_i \partial x_i$

Proposition (De Bie et al., Trans Amer Math Soc 2012)
For $A \subset [n]$, the operators $D_A$ and $x_A$ generate the Lie superalgebra $\mathfrak{osp}(1|2)$ with defining relations

\[
\{x_A, D_A\} = -2(\mathbb{E}_A + \gamma_A), \quad [D_A, \mathbb{E}_A + \gamma_A] = D_A, \quad [\mathbb{E}_A + \gamma_A, x_A] = x_A
\]

where $[x, y] = xy - yx$ stands for the commutator and where

\[
\gamma_A = \frac{|A|}{2} + \sum_{i \in A} \mu_i.
\]
sCasimir and spherical Dirac-Dunkl operators

- From $\mathfrak{osp}(1,2)$ symmetry, consider the sCasimir operator

$$ S_A = \frac{1}{2} ([x_A, D_A] - 1) $$

has the property

$$ \{S_A, D_A\} = 0 \quad \{S_A, x_A\} = 0 $$

- Spherical Dirac-Dunkl operators

$$ \Gamma_{[n]} = S_{[n]} \prod_{i=1}^{n} r_i \quad \Gamma_A = S_A = \prod_{i \in A} r_i $$

- Have the property

$$ [\Gamma_A, D_A] = 0 \quad [\Gamma_A, x_A] = 0 $$

- Also commute with $E_{[n]}$ (hence well-defined action on $S^{n-1}$)
Dunkl monogenics

- Let $\mathcal{P}(\mathbb{R}^n) = \mathbb{R}[x_1, \ldots, x_n]$ ring of polynomials
- Then $\mathcal{P}(\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k(\mathbb{R}^n)$ where $\mathcal{P}_k(\mathbb{R}^n)$ is space of homogeneous polynomials of degree $k$
- Let $V$ be $\text{Cl}_n$-module (fixed once and for all)
- Space of monogenics $\mathcal{M}_k(\mathbb{R}^n; V)$ of degree $k$ defined as

$$\mathcal{M}_k(\mathbb{R}^n; V) = \ker D \bigcap (\mathcal{P}_k(\mathbb{R}^n) \otimes V)$$


One has the decomposition

$$\mathcal{P}_k(\mathbb{R}^n) \otimes V = \bigoplus_{j=0}^{k} x_j^j \mathcal{M}_{k-j}(\mathbb{R}^n; V)$$
The Dirac-Dunkl equation on $S^{n-1}$

**Proposition**

Let $\Psi_k \in \mathcal{M}_k(\mathbb{R}^n; V)$ be a monogenic of degree $k$, then $\Psi_k$ satisfies the Dirac–Dunkl equation

$$\Gamma[n] \Psi_k = (-1)^k (k + \gamma[n] - 1/2) \Psi_k \quad (\star)$$

where $\gamma[n] = \sum_{i=1}^{n} \mu_i + \frac{n}{2}$

**Proof.**

Stems directly from $\mathfrak{osp}(1,2)$ symmetry

$$\Gamma[n] \Psi_k = \frac{(-1)^k}{2} \left( x[n]\sigma[n] - \sigma[n]x[n] - 1 \right) \Psi_k = \frac{(-1)^k}{2} \left( -\sigma[n]x[n] - 1 \right) \Psi_k$$

$$= \frac{(-1)^k}{2} \left( -x[n]\sigma[n] - \sigma[n]x[n] - 1 \right) \Psi_k = \frac{(-1)^{k+1}}{2} \left( \{x[n],\sigma[n]\} + 1 \right) \Psi_k$$

$$= \frac{(-1)^{k+1}}{2} \left( -2(\mathbb{E}[n] + \gamma[n]) + 1 \right) \Psi_k = (-1)^k (k + \gamma[n] - 1/2) \Psi_k,$$
Goals

- Determine the symmetries of (★)
- Investigate the corresponding symmetry algebra
- Determine wavefunctions $\Psi_k$
- Describe the action of symmetries on the wavefunctions
Symmetries of the Dirac-Dunkl equation on $S^{n-1}$

- We construct joint symmetries of Dirac-Dunkl ($D$) and spherical Dirac-Dunkl ($\Gamma_{[n]}$) operators
- Use the nested $\mathfrak{osp}(1,2)$ sCasimirs for all $A \subset [n]$

**Lemma**

For $A \subset [n]$, the operator $\Gamma_A$ satisfies

1. $[\Gamma_A, D] = [\Gamma_A, x] = 0$
2. $[\Gamma_A, \Gamma_{[n]}] = 0$

*Proof follows from the fact that $\Gamma_A$ commute with $D_A$, $x_A$ and act only in $x_i$ for $x_i \in A$*

- Hence $\Gamma_A$ for all $A \subset [n]$ are symmetries
- For $A \neq B$, clearly $\Gamma_A, \Gamma_B$ will not commute in general

**Remark:**

$$\Gamma_\emptyset = -\frac{1}{2}, \quad \Gamma_{\{i\}} = \mu_i$$
Some formulas

For the sake of being explicit

\[
\Gamma_A = \left( \sum_{\{i,j\} \subset A} M_{ij} + \frac{|A| - 1}{2} + \sum_{k \in A} \mu_k r_k \right) \prod_{i \in A} r_i
\]

\(M_{ij}\) reads

\[
M_{ij} = e_i e_j (x_j T_j - x_i T_i)
\]

- \(\Gamma_A\) generate symmetry algebra of the Dirac-Dunkl equation (⋆)
- Algebra is determined by the commutation relations of \(\Gamma\)'s
Proposition (De Bie, Genest, Vinet (2015))

The symmetries $\Gamma_A$ with $A \subset [n]$ of the Dirac–Dunkl and spherical Dirac–Dunkl operators satisfy the anticommutation relations

$$\{\Gamma_A, \Gamma_B\} = \Gamma_{(A \cup B) \setminus (A \cap B)} + 2 \Gamma_{A \cap B} \Gamma_{A \cup B} + 2 \Gamma_{A \setminus (A \cap B)} \Gamma_{B \setminus (A \cap B)}.$$

Proof is by direct calculations, and multiple inductions on the cardinality of the involved sets.

- Call this algebra $\mathcal{A}_n$
- Clearly $\mathcal{A}_n \subset \mathcal{A}_{n+1}$
- Can be show that all $\Gamma_{\{i,j\}}$ are sufficient generating set
  Unusual algebra! Let’s look at the $n = 3$ case.
The $n = 3$ case: The Bannai-Ito algebra

- $\mathcal{A}_n$ is higher rank generalization of Bannai-Ito algebra
- Take $n = 3$, symmetry algebra of $\Gamma_{[3]}$ is generated by
  \[ K_3 = \Gamma_{\{1,2\}} \quad K_1 = \Gamma_{\{2,3\}} \quad K_2 = \Gamma_{\{1,3\}} \]
- Commutation relations are
  \[
  \{K_1, K_2\} = K_3 + \omega_3 \quad \{K_2, K_3\} = K_1 + \omega_1 \quad \{K_3, K_1\} = K_2 + \omega_2
  \]
  where $\omega_1, \omega_2, \omega_3$ are given by
  \[
  \omega_1 = 2\mu_1 \Gamma_{[3]} + 2\mu_2 \mu_3 \quad \omega_2 = 2\mu_2 \Gamma_{[3]} + 2\mu_1 \mu_3 \\
  \omega_3 = 2\mu_3 \Gamma_{[3]} + 2\mu_1 \mu_2.
  \]
- $\omega_i$ are central elements, but on $\mathcal{M}_k(\mathbb{R}^3;V)$ they become numbers
- Structure associated to bispectrality of BI polynomials
- $n = 3$ case detailed in [De Bie, Genest, Vinet Comm. Math (2016)]
Abelian subalgebras

- Algebra $\mathcal{A}_n$ has an important (maximal) Abelian subalgebra $\mathcal{Y}_n$

  $$\mathcal{Y}_n = \langle \Gamma_{[2]}, \Gamma_{[3]}, \ldots, \Gamma_{[n-1]} \rangle$$

- Thus $\mathcal{A}_n$ has rank $(n - 2)$

- Non-unique, indeed $\mathcal{Z}_n$ is also (maximal) Abelian subalgebra

  $$\mathcal{Z}_n = \langle \Gamma_{\{2,3\}}, \Gamma_{\{2,3,4\}}, \ldots, \Gamma_{\{2,3,\ldots,n\}} \rangle$$

- Need to understand the wavefunctions $\Psi_k$ and action of $\Gamma_A$’s

- We’ll need a generalization of a construction known in Clifford analysis as the Cauchy-Kovalevskaia extension
Wavefunctions and the CK isomorphism

- We construct a basis of orthogonal wavefunctions $\Psi_k$ which are solutions of the Dirac-Dunkl equation on $S^{n-1}$

$$\Gamma_{[n]} \Psi_k = (-1)^k (k + \gamma_{[n]} - 1/2) \Psi_k \quad (\star)$$

- Key: there is an isomorphism $\text{CK}^\mu_j : \mathcal{P}_k(\mathbb{R}^{j-1}) \otimes V \rightarrow \mathcal{M}_k(\mathbb{R}^j; V)$

- This isomorphism $\text{CK}^\mu_j$ is explicit


The isomorphism between $\mathcal{P}_k(\mathbb{R}^{j-1}) \otimes V$ and $\mathcal{M}_k(\mathbb{R}^j; V)$ denoted by $\text{CK}^\mu_j$ is explicitly defined by the operator

$$\text{CK}^\mu_j =$$

$$\Gamma(\mu_j + 1/2) \left[ \bar{I}_{\mu_j - 1/2} (x_j D_{[j-1]}) + \frac{1}{2} e_j x_j D_{[j-1]} \bar{I}_{\mu_j + 1/2} (x_j D_{[j-1]}) \right],$$

with $\bar{I}_\alpha(x) = (\frac{2}{x})^\alpha I_\alpha(x)$ and $I_\alpha(x)$ the modified Bessel functions.
Constructive: one considers a power series in the operators $x_j$ and $D_{[j-1]}$ and solves for the coefficients to ensure that the result is in $\mathcal{M}_k(\mathbb{R}^n; V)$.

More explicitly

$$
\mathbf{CK}^{\mu j}_{x_j}(p) = \sum_{\ell=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{\Gamma(\mu_j + 1/2)}{2^{2\ell} \ell! \Gamma(\ell + \mu_j + 1/2)} x_j^{2\ell} D_{[j-1]}^{2\ell} p
$$

$$
+ \frac{e_j x_j D_{[j]}}{2} \sum_{\ell=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{\Gamma(\mu_j + 1/2)}{2^{2\ell} \ell! \Gamma(\ell + \mu_j + 3/2)} x_j^{2\ell} D_{[j-1]}^{2\ell} p.
$$

with $p \in \mathcal{P}_k(\mathbb{R}^n) \otimes V$.

How is this helpful?

We can combine this theorem with the (Fischer) decomposition presented earlier

$$
\mathcal{P}_k(\mathbb{R}^n) \otimes V = \bigoplus_{j=0}^{k} x^j \mathcal{M}_{k-j}(\mathbb{R}^n; V)
$$

to construct a basis.
Let \( \{v_s\} \) for \( s = 1, \ldots, \dim V \) be a basis for representation space \( V \) of \( Cl_n \).

Combining \( \text{CK}_{x_j}^{\mu_j} \), one can write

\[
\mathcal{M}_k(\mathbb{R}^n; V) \equiv \text{CK}_{x_n}^{\mu_n} [\mathcal{P}_k(\mathbb{R}^{n-1}) \otimes V] \equiv \text{CK}_{x_n}^{\mu_n} \left[ \bigoplus_{j=0}^{k} x_{[n-1]}^{k-j} \mathcal{M}_j(\mathbb{R}^{n-1}; V) \right]
\]

\[
\cong \text{CK}_{x_n}^{\mu_n} \left[ \bigoplus_{j=0}^{k} x_{[n-1]}^{k-j} \text{CK}_{x_{n-1}}^{\mu_{n-1}} [\mathcal{P}_j(\mathbb{R}^{n-2}) \otimes V] \right]
\]

\[
\cong \text{CK}_{x_n}^{\mu_n} \left[ \bigoplus_{j=0}^{k} x_{[n-1]}^{k-j} \text{CK}_{x_{n-1}}^{\mu_{n-1}} \left[ \bigoplus_{\ell=0}^{j} x_{[n-2]}^{j-\ell} \mathcal{M}_\ell(\mathbb{R}^{n-2}; V) \right] \right] \cong \cdots
\]

We can go down the tower until we reach the space of Clifford-valued homogeneous polynomials of a certain degree \( j_1 \) in one variable, which is spanned by \( x_1^{j_1} v_s \).
Wavefunctions

Proposition (De Bie, G., Vinet (2015))

Let \( \mathbf{j} \) be defined as \( \mathbf{j} = (j_1, j_2, \ldots, j_{n-2}, j_{n-1} = k - \sum_{i=1}^{n-2} j_i) \) where \( j_1, \ldots, j_{n-2} \) are non-negative integers such that \( \sum_{i=1}^{n-2} j_i \leq k \). Consider the set of functions \( \Psi^s_j(x_1, \ldots, x_n) \) defined by

\[
\Psi^s_j(x_1, \ldots, x_n) = \text{CK}_{x_n}^{\mu_n} \left[ x_1^{j_1-1} \text{CK}_{x_3}^{\mu_3} \left[ x_2^{j_2-1} \text{CK}_{x_2}^{\mu_2} \left[ x_1^{j_1-1} \right] \right] \right] v_s,
\]

with \( s \in I \). Then the functions \( \Psi^s_j \) form a basis for the space \( \mathcal{M}_k(\mathbb{R}^n; V) \) of \( k \)-homogeneous Dunkl monogenics.

- Since \( \text{CK}_{x_j}^{\mu_j} \) is known, these wavefunctions can be made explicit
- Long calculations involved, but relatively straightforward
Wavefunctions

Some notation

$$|\mathbf{j}_\ell| = j_1 + j_2 + \cdots + j_\ell \quad ||x_{[j]}||^2 = x_1^2 + \cdots + x_j^2$$

Proposition (De Bie, G., Vinet (2015))

The basis functions $\Psi_{j}^{s}$ have the expression

$$\Psi_{j}^{s}(x_1, \ldots, x_n) = \left( \prod_{\ell=n}^{3} \right) Q_{j_{\ell-1}}(x_{\ell}, x_{[\ell-1]}) m_{j_1}(x_2, x_1) v_s,$$
$$Q_{j\ell-1}(x_{\ell}, x_{[\ell-1]}) = \frac{\beta! \|x_{[\ell]}\|^{2\beta}}{\left(\mu_{\ell} + 1/2\right)^{\beta}}$$

$$\times \left\{ \begin{array}{ll}
P^{(\jmath_{\ell-2} + \gamma_{\ell-1} - 1, \mu_{\ell} - 1/2)}_{\beta} \left( x_{\ell}^2 - \|x_{[\ell-1]}\|^2 \right) \\
- e_{\ell} x_{\ell} \frac{x_{[\ell-1]}}{\|x_{[\ell]}\|^2} P^{(\jmath_{\ell-2} + \gamma_{\ell-1}, \mu_{\ell} + 1/2)}_{\beta-1} \left( x_{\ell}^2 - \|x_{[\ell-1]}\|^2 \right) \\
-x_{[\ell-1]} P^{(\jmath_{\ell-2} + \gamma_{\ell-1}, \mu_{\ell} - 1/2)}_{\beta} \left( x_{\ell}^2 - \|x_{[\ell-1]}\|^2 \right) \\
- e_{\ell} x_{\ell} \left( \frac{\beta + \jmath_{\ell-2} + \gamma_{\ell-1}}{\beta + \mu_{\ell} + 1/2} \right) P^{(\jmath_{\ell-2} + \gamma_{\ell-1} - 1, \mu_{\ell} + 1/2)}_{\beta} \left( x_{\ell}^2 - \|x_{[\ell-1]}\|^2 \right) \end{array} \right\}$$

$$j\ell-1 = 2\beta$$

$$j\ell-1 = 2\beta + 1$$
\[ m_{j_1}(x_2, x_1) = \frac{(-1)^\beta \beta! (x_1^2 + x_2^2)^\beta}{(\mu_2 + 1/2)^\beta} \]

\[ \begin{align*}
&\left\{ \begin{array}{l}
P^{(\mu_1-1/2,\mu_2-1/2)}_{\beta} \left( \frac{x_2^2-x_1^2}{x_1^2+x_2^2} \right) \\
- e_2e_1 x_2 x_1 \ P^{(\mu_1+1/2,\mu_2+1/2)}_{\beta-1} \left( \frac{x_2^2-x_1^2}{x_1^2+x_2^2} \right), \\
x_1 P^{(\mu_1+1/2,\mu_2-1/2)}_{\beta} \left( \frac{x_2^2-x_1^2}{x_1^2+x_2^2} \right) \\
+ (\frac{\beta + \mu_1 + 1/2}{\beta + \mu_2 + 1/2}) e_2 e_1 x_2 P^{(\mu_1-1/2,\mu_2+1/2)}_{\beta} \left( \frac{x_2^2-x_1^2}{x_1^2+x_2^2} \right)
\end{array} \right. \\
\times \begin{cases} j_1 = 2\beta \\
\end{cases}
&\begin{cases} j_1 = 2\beta + 1 \\
\end{cases}
\end{align*} \]

- \( P^{(\alpha,\beta)}(x) \) are the Jacobi polynomials
- \( (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \) stands for the Pochhammer symbol
Algebra $\mathcal{Y}_n$ and orthogonality

Wavefunctions $\Psi^s_j(x_1,\ldots,x_n)$ are joint eigenfunctions of the commutative subalgebra $\mathcal{Y}_n$

**Proposition (De Bie, G., Vinet (2015))**

The wavefunctions $\Psi^s_j \in \mathcal{M}_k(\mathbb{R}^n; V)$ satisfy

$$\Gamma_{[\ell]} \Psi^s_j(x_1,\ldots,x_n) = \lambda_\ell(j) \Psi^s_j(x_1,\ldots,x_n),$$

where the eigenvalues are given by

$$\lambda_\ell(j) = (-1)^{|j|_\ell-1}(|j|_{\ell-1} + \gamma_{[\ell]} - 1/2).$$

- Up to a normalization factor, one has

$$\int_{\mathbb{S}^{n-1}} |x_1|^{2\mu_1}dx_1 \cdots |x_n|^{2\mu_n}dx_n \left[ \Psi^s'_{j'}(x_1,\ldots,x_n) \right]^* \cdot \Psi^s_j(x_1,\ldots,x_n) = \delta_{jj'} \delta_{ss'},$$

where $*$ is complex conjugation and $\cdot$ stands for an appropriately defined inner product on $V$; one has also $e_i^* = -e_i$. 
Algebra $\mathcal{Z}_n$ and Bannai-Ito polynomials

- From the common eigenfunctions $\Psi_j^s(x_1,\ldots,x_n)$ of $\mathcal{Y}_n$, one can easily construct a set of joint eigenfunctions for the Abelian subalgebra $\mathcal{Z}_n$.
- Let $\pi = (12 \cdots n)$ be the cyclic permutation acting on $x_1,\ldots,x_n$, $e_1,\ldots,e_n$ and $\mu_1,\ldots,\mu_n$, then common eigenfunctions $\Phi_j^s(x_1,\ldots,x_n)$ of $\mathcal{Z}_n$

$$\Phi_j^s(x_1,\ldots,x_n) = \pi \Psi_j^s(x_1,\ldots,x_n) \quad \pi = (123 \cdots n),$$

- $\Phi_j^s(x_1,\ldots,x_n)$ also form a basis for $\mathcal{M}_k(\mathbb{R}^n;V)$
- Since both basis are orthonormal and finite dimensional, there exist a unitary transformation between the two bases

**Conjecture**

The matrix elements of the intertwining operator between the basis functions $\Psi_j^s(x_1,\ldots,x_n)$ and $\Phi_j^s(x_1,\ldots,x_n)$ are expressed in terms of orthogonal polynomials which could be taken to define the multivariate extension of the Bannai–Ito polynomials.
Representations of $\mathcal{A}_n$

- It is clear that functions $\Psi_j^s(x_1,\ldots,x_n)$ will support representations of $\mathcal{A}_n$
- To specify these representations, it is needed to find the action of the operators $\Gamma_A$ on these functions
- This can be done via the introduction of raising/lowering operators

\[
K^\pm_\ell = (\Gamma_{\{\ell+1,\ell+2\}} \pm \Gamma_{[\ell+2] \setminus \{\ell+1\}})(\Gamma_{[\ell+1]} \mp 1/2) - (\Gamma_{[\ell+2]} \pm \Gamma_{[\ell+2]})(\Gamma_{[\ell]} \pm \Gamma_{[\ell+1]}).
\]

for $\ell \in [n-2]$ and observing that

\[
[K^\pm_\ell, \Gamma_{[j]}] = 0 \quad j \neq \ell + 1
\]

Conversely when $j = \ell + 1$ one has

\[
\{K^\pm_\ell, \Gamma_{[\ell+1]}\} = \pm K^\pm_\ell
\]
Final remarks

- There exists an equivalent scalar model that does not involve the Clifford algebra.
- Much work to be done to understand $A_n$ further (representations, etc.)
- The conjecture should be checked!
- Opens the door for higher rank versions of the Racah algebra, which would describe the bispectrality of the multivariate Racah polynomials.
- One could also investigate other root systems.