Locating pairs of vertices on Hamiltonian cycles

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joint work with Hao Li, Qiang Sun

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Outline

1. The Enomoto conjecture
2. Szemerédi’s Regularity Lemma
3. Sketch of the proof
4. The Faudree-Li conjecture
5. Further works
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2. Szemerédi’s Regularity Lemma
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Hamiltonian cycle

- **Graph**: $G = (V(G), E(G))$.
- **Hamiltonian cycle**: a cycle visits each vertex exactly once.

- **Hamiltonian problem**: determining whether a Hamiltonian cycle exists in a given graph.
Dirac’s theorem

**Theorem (Dirac, 1952)**

Let $G$ be a graph with $n \geq 3$ vertices. If $\delta(G) \geq \frac{n}{2}$, then $G$ is Hamiltonian.

**Theorem (Ore, 1960)**

Let $G$ be a graph with $n \geq 3$ vertices. If $\sigma_2(G) \geq n$ (i.e. $\deg(x) + \deg(y) \geq n$ for any pair of nonadjacent vertices $x$ and $y$ in $G$), then $G$ is Hamiltonian.
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Theorem (Kaneko and Yoshimoto, 2001)

Let $G$ be a graph of order $n$ with $\delta(G) \geq \frac{n}{2}$, and let $d$ be a positive integer such that $d \leq \frac{n}{4}$. Then, for any vertex subset $S$ with $|S| \leq \frac{n}{2d}$, there is a Hamiltonian cycle $C$ such that $\text{dist}_C(u, v) \geq d$ for any $u, v \in S$. 
Sárközy and Selkow showed that almost all of the distances between successive pairs of $S$ can be specified almost exactly.

**Theorem (Sárközy and Selkow, 2008)**

There are $\omega, n_0 > 0$ such that if $G$ is a graph with $\delta(G) \geq \frac{n}{2}$ on $n \geq n_0$ vertices, $d$ is an arbitrary integer with $3 \leq d \leq \frac{\omega n}{2}$ and $S$ is an arbitrary subset of $V(G)$ with $2 \leq |S| = k \leq \frac{\omega n}{2}$, then for every sequence of integers with $3 \leq d_i \leq d$, and $1 \leq i \leq k - 1$, there is a Hamiltonian cycle $C$ of $G$ and an ordering of the vertices of $S$, $a_1, a_2, ..., a_k$, such that the vertices of $S$ are encountered in this order on $C$ and we have $|\text{dist}_C(a_i, a_{i+1}) - d_i| \leq 1$, for all but one $1 \leq i \leq k - 1$. 
Locating pairs of vertices on Hamiltonian cycles

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Conjecture (Enomoto)

If $G$ is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices $x, y$ in $G$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = \left\lfloor \frac{n}{2} \right\rfloor$.

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If $G$ is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices $x, y$ in $G$ and any integer $2 \leq k \leq \frac{n}{2}$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = k$. 
The degree condition is sharp.

Example 1: there is no Hamiltonian cycle such that $x$ and $y$ have distance $\frac{n}{2}$.

Figure: $2K_{\frac{n}{2}-1} + K_2$
The degree condition is sharp.

Example 2: $x$ and $y$ will be at distance $\frac{n}{2}$ in any Hamiltonian cycle of the graph.

Figure: $2K_{\frac{n}{2}-1} + K_2$
Theorem (Faudree and Li, 2012)

If $p$ is a positive integer with $2 \leq p \leq \frac{n}{2}$ and $G$ is a graph of order $n$ with $\delta(G) \geq \frac{n+p}{2}$, then for any pair of vertices $x$ and $y$ in $G$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = k$ for any $2 \leq k \leq p$.

Corollary (Faudree and Li, 2012)

If $G$ is a graph of order $n$ with $\delta(G) \geq \lfloor \frac{3n}{4} \rfloor$, then for any pair of vertices $x$ and $y$ of $G$ and any positive integer $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = k$. 
Locating pairs of vertices on Hamiltonian cycles

**Theorem (Faudree and Li, 2012)**

If $p$ is a positive integer with $2 \leq p \leq \frac{n}{2}$ and $G$ is a graph of order $n$ with $\delta(G) \geq \frac{n+p}{2}$, then for any pair of vertices $x$ and $y$ in $G$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = k$ for any $2 \leq k \leq p$.

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Our result

Theorem (He, Li and Sun, 2015)

There exists a positive integer $n_0$ such that for all $n \geq n_0$, if $G$ is a graph of order $n$ with $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices $x, y$ in $G$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = \left\lfloor \frac{n}{2} \right\rfloor$. 
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Density: Let $G$ be a graph, for any two disjoint vertex sets $X$ and $Y$ of $G$. The density of the pair $(X, Y)$ is the ratio 
$$d'(X, Y) := \frac{e(X, Y)}{|X||Y|}.$$ 

$\epsilon$-regularity: Let $\epsilon > 0$, the pair $(X, Y)$ is called $\epsilon$-regular if for every $A \subseteq X$ and $B \subseteq Y$ such that $|A| > \epsilon |X|$ and $|B| > \epsilon |Y|$ we have $|d(A, B) - d(X, Y)| < \epsilon$.

Super-regularity: Let $\delta > 0$, the pair $(X, Y)$ is called $(\epsilon, \delta)$-super-regular if it is $\epsilon$-regular, $\deg_Y(x) > \delta |Y|$ for all $x \in X$ and $\deg_X(y) > \delta |X|$ for all $y \in Y$. 

Lemma (Regularity Lemma-Degree Form)

For every $\epsilon > 0$ and every integer $m_0$ there is an $M_0 = M_0(\epsilon, m_0)$ such that if $G = (V, E)$ is any graph on at least $M_0$ vertices and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set $V$ into $l + 1$ clusters $V_0, V_1, \ldots, V_l$, and there is a subgraph $G' = (V, E')$ with the following properties:

1. $m_0 \leq l \leq M_0$;
2. $|V_0| \leq \epsilon|V|$, and $V_i$ ($1 \leq i \leq l$) are of the same size $L$;
3. $\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)|V|$ for all $v \in V$;
4. $G'[V_i] = \emptyset$ (i.e. $V_i$ is an independant set in $G'$) for all $i$;
5. each pair $(V_i, V_j)$, $1 \leq i < j \leq l$, is $\epsilon$-regular, each with a density $0$ or exceeding $d$. 

Weihua He (GDUT)
Regularity lemma

G → G'
- **Arithmetic progression of length** $k$: a set of integers of the form

\[ \{a, a + d, a + 2d, \ldots, a + (k - 1)d\} \].

**Conjecture (Erdős, Turán, 1936)**

If the sum of reciprocals of a set $A$ of positive integers diverges (i.e. $\sum_{n \in A} \frac{1}{n} = \infty$), then that $A$ contains arbitrarily long arithmetic progressions.

**Theorem (Szemerédi’s Theorem, 1975)**

For any integer $k \geq 1$ and $\delta > 0$ there is an integer $N = N(k, \delta)$ such that any subset $S \subseteq \{1, \ldots, N\}$ with $|S| \geq \delta N$ contains an arithmetic progression of length $k$. 
Conjecture (Nash-Williams, 1971)

Let $G$ be a $d$-regular graph on at most $2d$ vertices. Then $G$ contains $\left\lfloor \frac{d}{2} \right\rfloor$ edge-disjoint Hamiltonian cycles.

Theorem (Christofides, Kühn and Osthus, 2012)

For every $\alpha > 0$ there is an integer $n_0$ so that every $d$-regular graph on $n \geq n_0$ vertices with $d \geq \left(\frac{1}{2} + \alpha\right)n$ contains at least $\frac{d-\alpha n}{2}$ edge-disjoint Hamiltonian cycles.
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A vertex-transitive graph: a graph $G$ such that, given any two vertices $v_1$ and $v_2$ of $G$, there is some automorphism $f : V(G) \rightarrow V(G)$ such that $f(v_1) = v_2$.

Conjecture (Lovász, 1970)

Every connected vertex-transitive graph has a Hamiltonian path.

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For every $\alpha > 0$ there exists an $n_0$ such that every connected vertex-transitive graph on $n \geq n_0$ vertices with minimum degree at least $\alpha n$ contains a Hamiltonian cycle.
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Preparation of the proof

Theorem (He, Li and Sun, 2015)

There exists a positive integer $n_0$ such that for all $n \geq n_0$, if $G$ is a graph of order $n$ with $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices $x, y$ in $G$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = \lfloor \frac{n}{2} \rfloor$.

- Only need to consider the graphs with even order.
- Suppose $0 < \epsilon \ll d \ll \alpha \ll 1$, and $n$ is sufficiently large.
- A balanced partition of $V(G)$ into $V_1$ and $V_2$ is a partition of $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = \frac{n}{2}$.
  - **Extremal Case 1**: There exists a balanced partition of $V(G)$ into $V_1$ and $V_2$ such that the density $d(V_1, V_2) \geq 1 - \alpha$.
  - **Extremal Case 2**: There exists a balanced partition of $V(G)$ into $V_1$ and $V_2$ such that the density $d(V_1, V_2) \leq \alpha$. 
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Non-extremal case

Step 1: constructing a Hamiltonian cycle in the reduced graph

Let $G$ be a graph not in either of the extremal cases. We apply the Regularity Lemma to $G$.

**Reduced graph** $R$: the vertices of $R$ are $r_1, r_2, ..., r_l$, and there is an edge between $r_i$ and $r_j$ if the pair $(V_i, V_j)$ is $\epsilon$-regular in $G'$ with density exceeding $d$.

- $R$ inherits the minimum degree condition: $\delta(R) \geq (\frac{1}{2} - 2d)l$.
- $R$ is a Hamiltonian graph.
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  - $R$ inherits the minimum degree condition: $\delta(R) \geq \left(\frac{1}{2} - 2d\right)l$.
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Non-extremal case
Step 2: constructing paths to connect clusters

- By the Hamiltonian cycle in $R$, we find a perfect matching in $R$. Denote the clusters by $X_i, Y_i$ according to the matching. $(X_i, Y_i)$ is called a pair of clusters.

- Construct paths $P_i$’s and $Q_i$’s to connect different pairs of clusters and to include $x, y$. 
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Figure: Construction of $P_i$'s and $Q_i$'s.
Non-extremal case

Step 3: Extending the paths by all the vertices of $V_0$

- Deal with the vertices of $V_0$ pair by pair.

**Figure:** Insert $u, v \in V_0$ to $Q_i$'s.
Non-extremal case

Step 4: constructing the desired Hamiltonian cycle

Lemma (Blow-up Lemma-Bipartite Version)

For every $\delta, \Delta > 0$, there exists an $\epsilon = \epsilon(\delta, \Delta) > 0$ such that the following holds. Let $(X, Y)$ be an $(\epsilon, \delta)$-super-regular pair with $|X| = |Y| = N$. If a bipartite graph $H$ with $\Delta(H) \leq \Delta$ can be embedded in $K_{N,N}$ by a function $\phi$, then $H$ can be embedded in $(X, Y)$.

Construct paths $W_i^1$'s and $W_i^2$'s in each pair of clusters by Blow-up lemma and make sure $x$ and $y$ have distance $\frac{n}{2}$ on this cycle.
Extremal Case 1: There exists a balanced partition of $V(G)$ into $V_1$ and $V_2$ such that the density $d(V_1, V_2) \geq 1 - \alpha$.

Lemma

If $G$ is in extremal case 1, then $G$ contains a balanced spanning bipartite subgraph $G^*$ with parts $U_1$, $U_2$ and $G^*$ has the following properties:

(a) there is a vertex set $W$ such that there exist vertex-disjoint 2-paths (paths of length two) in $G^*$ with the vertices of $W$ as the middle vertices (not the end vertices) in each 2-path and $|W| \leq \alpha_2 n$;

(b) $\deg_{G^*}(v) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$ for all $v \notin W$. 
Extremal case 1

The proof has some sub-cases discussions depending on the position of $x, y$ and the parity of $\frac{n}{2}$. And the Blow-up lemma is the main tool.

Figure: Extremal case 1.
Extremal Case 2: There exists a balanced partition of $V(G)$ into $V_1$ and $V_2$ such that the density $d(V_1, V_2) \leq \alpha$.

Lemma

If $G$ is in extremal case 2, then $V(G)$ can be partitioned into two balanced parts $U_1$ and $U_2$ such that

(a) there is a set $W_1 \subseteq U_1$ (resp. $W_2 \subseteq U_2$) such that there exist vertex-disjoint 2-paths in $G[U_1]$ (resp. $G[U_2]$) with the vertices of $W_1$ (resp. $W_2$) as the middle vertices in each 2-path and $|W_1| \leq \alpha_2 \frac{n}{2}$ (resp. $|W_2| \leq \alpha_2 \frac{n}{2}$);

(b) $\deg_{G[U_1]}(u) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$ for all $u \in U_1 - W_1$ and $\deg_{G[U_2]}(v) \geq (1 - \alpha_1 - 2\alpha_2) \frac{n}{2}$ for all $v \in U_2 - W_2$. 
Extremal case 2

The proof has some sub-cases discussions depending on the position of $x$ and $y$.

Figure: Extremal case 2.
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The Faudree-Li conjecture

Conjecture (Faudree and Li, 2012)

If $G$ is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices $x, y$ in $G$ and any integer $2 \leq k \leq \frac{n}{2}$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = k$.

Theorem (Faudree and Li, 2012)

If $G$ is a graph of order $n \geq 3$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices $x, y$ in $G$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = 2$ and a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = 3$. 
The Faudree-Li conjecture

Theorem (Faudree, Lehel and Yoshimoto, 2014)

Let $k \geq 2$ be a fixed positive integer. If $G$ is a graph of order $n \geq 6k$ and $\delta(G) \geq \frac{n}{2} + 1$, then for any pair of vertices $x, y$ in $G$, there is a Hamiltonian cycle $C$ of $G$ such that $\text{dist}_C(x, y) = k$. 
Our approach also works for the Faudree-Li conjecture.

**Theorem (He, Li and Sun, 2015)**

There exists a positive integer \( n_0 \) such that for all \( n \geq n_0 \), if \( G \) is a graph of order \( n \) with \( \delta(G) \geq \frac{n}{2} + 1 \), then for any pair of vertices \( x, y \) in \( G \) and any integer \( 2 \leq k \leq \frac{n}{2} \), there is a Hamiltonian cycle \( C \) of \( G \) such that \( \text{dist}_C(x, y) = k \).
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Further works

- To avoid using Szemerédi’s regularity lemma?
- To locate more vertices ($\geq 3$) on Hamiltonian cycles with precise distances?
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Thank you!