Hopf Algebras: A Basic Introduction
(intended for undergraduate students)

Kyoung-Tark Kim
kyoungtarkkim@sjtu.edu.cn
Shanghai Jiao Tong University

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Based on the following textbooks:

**Moss Eisenberg Sweedler,**

Hopf algebras,

*Mathematics Lecture Note Series, W. A. Benjamin, 1969*

**S. Dăscălescu, C. Năstăsescu, Ş. Raianu,**

Hopf algebras: an introduction,

*Monographs and Textbooks in Pure and Applied Mathematics 235, Marcel Dekker, 2001*

**Tonny Albert Springer,**

Linear algebraic groups,

In this presentation, $K$ denotes a field, and all tensor products are over $K$, e.g., $V \otimes W = V \otimes_K W$.

All rings and associative algebras are assumed to have identity.
Chapter 1.
Basic Definitions, Notions, and Examples
Definition of (associative) algebras over $\mathbb{K}$

There are many equivalent definitions for an (associative) algebra $A$ over $\mathbb{K}$:

- $A$ is a ring together with a ring homomorphism $\mathbb{K} \rightarrow A$ whose image is in the center of $A$.

- $A$ is a $\mathbb{K}$-vector space together with a $\mathbb{K}$-bilinear operation $A \times A \rightarrow A$ such that $(xy)z = x(yz)$, $\forall x, y, z \in A$, in which $A$ has multiplicative identity.
What is a 'good' definition of algebras for us?

Among these equivalent ones we adopt the following (next page) definition of algebras over $\mathbb{K}$ because it can be easily dualizable.
Definition of (associative) algebras over $\mathbb{K}$, continued

$A$ is called an algebra over $\mathbb{K}$ if $A$ is a $\mathbb{K}$-vector space together with two $\mathbb{K}$-linear maps $M : A \otimes A \to A$ and $u : \mathbb{K} \to A$ such that

\[
\begin{align*}
A \otimes A \otimes A & \xrightarrow{\text{Id} \otimes M} A \otimes A \\
A \otimes A & \xrightarrow{M} A \\
A \otimes \mathbb{K} & \xrightarrow{\sim} A
\end{align*}
\]

commute, where $\text{Id} : A \to A$ is the identity map.

We call $M$ a \textit{product} and $u$ a \textit{unit}, because

$xy := M(x \otimes y)$ and $1_A := u(1_{\mathbb{K}})$ play role as a usual multiplication and identity in $A$. 
Dualizing

By *reversing* all the directions of the arrows,

we obtain the notion of coalgebras over $K$...
Definition of coalgebras (cogebras) over $\mathbb{K}$

A coalgebra $C$ over $\mathbb{K}$ is a $\mathbb{K}$-vector space together with two $\mathbb{K}$-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$ such that

\[
\begin{array}{ccc}
C \otimes C \otimes C & \xrightarrow{\text{Id} \otimes \Delta} & C \otimes C \\
\Delta \otimes \text{Id} & & \Delta \\
C \otimes C & \xleftarrow{\Delta} & C
\end{array}
\]  
\[
\begin{array}{ccc}
C \otimes C & \xrightarrow{\epsilon \otimes \text{Id}} & \mathbb{K} \otimes C \\
\text{Id} \otimes \epsilon & \Delta & \simeq \\
C \otimes \mathbb{K} & \xleftarrow{\simeq} & C
\end{array}
\]

commute.

We call $\Delta$ a \textit{coproduct} and $\epsilon$ a \textit{counit}.

The identity $(\text{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{Id}) \circ \Delta$ from the first diagram is referred to as the “\textit{coassociativity}”. 
Commutativity and Cocommutativity

- An algebra \((A, M, u)\) is said to be **commutative** if
  \[
  A \otimes A \xrightarrow{x \otimes y \mapsto y \otimes x} A \otimes A
  \]
  commutes.

- A coalgebra \((C, \Delta, \epsilon)\) is said to be **cocommutative** if
  \[
  C \otimes C \xrightarrow{x \otimes y \mapsto y \otimes x} C \otimes C
  \]
  commutes.
Examples of coalgebras (I)

Ex. 1. ‘Group-like coalgebra’

Let $S$ be a set and $V$ a $\mathbb{K}$-space with the set $S$ as basis.

Define $\Delta : V \rightarrow V \otimes V$ and $\epsilon : V \rightarrow \mathbb{K}$ by

$\Delta(s) := s \otimes s$ and $\epsilon(s) := 1$, $\forall s \in S$.

Then $V$ becomes a (cocomutative) coalgebra over $\mathbb{K}$.

Ex. 2. ‘Devided power coalgebra’

Let $D$ be a $\mathbb{K}$-vector space with a basis $\{d_m | m = 0, 1, 2, \ldots \}$.

Define $\Delta : D \rightarrow D \otimes D$ and $\epsilon : D \rightarrow \mathbb{K}$ by

$\Delta(d_m) := \sum_{k=0}^{m} d_k \otimes d_{m-k}$ and $\epsilon(d_m) := \delta_{0,m}$, $\forall m = 0, 1, 2, \ldots$.

Then $D$ becomes a (cocomutative) coalgebra.
Examples of coalgebras (II)

Ex. 3. ‘Matrix coalgebra’

Let \( \{ e_{ij} \}_{1 \leq i, j \leq n} \) be the canonical basis for \( M := \text{Mat}_n(\mathbb{K}) \).

Then \( M \) is a coalgebra if \( \Delta : M \to M \otimes M \) and \( \epsilon : M \to \mathbb{K} \) are

\[
\Delta(e_{ij}) := \sum_{k=1}^{n} e_{ik} \otimes e_{kj} \quad \text{and} \quad \epsilon(e_{ij}) := \delta_{ij}.
\]

Ex. 4. ‘Incidence coalgebra’

Let \((P, \leq)\) be a locally finite partially ordered set, i.e, for any \( x, y \in P \) with \( x \leq y \), the set \( \{ z | x \leq z \leq y \} \) is finite.

If \( V \) is a \( \mathbb{K} \)-vector space with \( \{(x, y) \in P \times P | x \leq y \} \) as basis,

\[
\Delta((x, y)) := \sum_{x \leq z \leq y} (x, z) \otimes (z, y), \quad \text{and} \quad \epsilon((x, y)) := \delta_{x, y},
\]

then \( V \) becomes a coalgebra.
Morphisms of algebras and coalgebras

- A $\mathbb{K}$-linear map $f : A \to B$ of algebras is a \textit{morphism} if

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
M_A & \downarrow & M_B \\
A & \xrightarrow{f} & B
\end{array}
\]

commute.

- A $\mathbb{K}$-linear map $g : C \to D$ of coalgebras is a \textit{morphism} if

\[
\begin{array}{ccc}
C & \xrightarrow{g} & D \\
\Delta_C & \downarrow & \Delta_D \\
C \otimes C & \xrightarrow{g \otimes g} & D \otimes D
\end{array}
\]

commute.
Generalized coassociativity

- In algebra $A$, we know the “generalized associativity”, e.g.,

$$(ab)((cd)((ef)g)) = a(b(((cd)e)(fg))) \quad \forall a, b, c, d, e, f, g \in A.$$  

- In coalgebra $(C, \Delta, \epsilon)$, put $\Delta_1 := \Delta$ and define recursively

$$\Delta_n : C \rightarrow C \otimes \cdots \otimes C$$

by $\Delta_n := (\Delta \otimes \text{Id} \otimes \cdots \otimes \text{Id}) \circ \Delta_{n-1}$.  

Then we have “generalized coassociativity”:

For any $n \geq 2$, $k \in \{1, \cdots, n-1\}$, and $p \in \{0, \cdots, n-k\}$,

$$\Delta_n = (\underbrace{\text{Id} \otimes \cdots \otimes \text{Id}}_{p \text{ times}} \otimes \Delta_k \otimes \underbrace{\text{Id} \otimes \cdots \otimes \text{Id}}_{n-k-p \text{ times}}) \circ \Delta_{n-k}$$ holds.
Product vs Coproduct

- We can view a product map as “law of composition”, i.e.,

\[ z := xy = M(x \otimes y). \]

The resulting quantity \( z = xy \) is more simple than \( x \) and \( y \) in the sense that the number of quantities decreases.

- However, a coproduct map is a “law of decomposition”, i.e.,

\[ \Delta(x) = \sum_{i,j} x_{1i} \otimes x_{2j}. \]

Usually, \( \Delta \) produces lots of resulting quantities \( x_{1i} \) and \( x_{2j} \), and hence we need many summation indices for them.
The sigma notation (a.k.a. Sweedler notation)

“WARNING!! The notation introduced in this section plays a key role in the sequel...”


For coproduct $\Delta$ or generalized coproduct $\Delta_n$, the sigma notation just supresses summation indices of resulting quantities.

For instance, if

$$\Delta(x) = \sum_{i,j} x_{1i} \otimes x_{2j} \quad \text{and} \quad \Delta_3(x) = \sum_{i,j,k,\ell} x_{1i} \otimes x_{2j} \otimes x_{3k} \otimes x_{4\ell},$$

then the sigma notation suggests to write the above equations as

$$\Delta(x) = \sum x_1 \otimes x_2 \quad \text{and} \quad \Delta_3(x) = \sum x_1 \otimes x_2 \otimes x_3 \otimes x_4.$$
Examples for use of the sigma notation

Let \((C, \Delta, \epsilon)\) be a coalgebra and \(x \in C\).

**Ex. 1.** The coassociativity \((\text{Id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{Id}) \circ \Delta = \Delta_2\) is

\[
\sum x_1 \otimes (x_2)_1 \otimes (x_2)_2 = \sum (x_1)_1 \otimes (x_1)_2 \otimes x_2 = \sum x_1 \otimes x_2 \otimes x_3.
\]

**Ex. 2.** The defining identity of the counit \(\epsilon\) is

\[
\sum \epsilon(x_1) \otimes x_2 = x = \sum x_1 \otimes \epsilon(x_2).
\]

**Ex. 3.** A \(\mathbb{K}\)-linear map \(g : C \rightarrow D\) is a coalgebra morphism iff

\[
\sum g(x_1) \otimes g(x_2) = \sum g(x)_1 \otimes g(x)_2 \quad \text{and} \quad \epsilon_C(x) = \epsilon_D(g(x)).
\]
Warm up practice

If \((C, \Delta, \epsilon)\) be a coalgebra, can you verify the following identities?

Exer. 1. \(\sum \epsilon(x_2) \otimes \Delta(x_1) = \Delta(x).\)

Exer. 2. \(\sum \Delta(x_2) \otimes \epsilon(x_1) = \Delta(x).\)

Exer. 3. \(\sum x_1 \otimes \epsilon(x_3) \otimes x_2 = \Delta(x).\)

Exer. 4. \(\sum x_1 \otimes x_3 \otimes \epsilon(x_2) = \Delta(x).\)

Exer. 5. \(\sum \epsilon(x_1) \otimes x_3 \otimes x_2 = \sum x_2 \otimes x_1.\)

Exer. 6. \(\sum \epsilon(x_1) \otimes \epsilon(x_3) \otimes x_2 = x.\)
Computation rule using the sigma notation

\((C, \Delta, \epsilon) : a\) coalgebra over \(\mathbb{K}\)

\(f : C \otimes \cdots \otimes C \to C : a\) \(\mathbb{K}\)-linear map

\(f : C \to C : the\ composition\ map\ \(C \xrightarrow{\Delta_n} C \otimes \cdots \otimes C \xrightarrow{f} C\).

\(g : C \otimes \cdots \otimes C \to C : a\) \(\mathbb{K}\)-linear map with \(k \geq n\)

\(g : C \otimes \cdots \otimes C \to C : a\) \(\mathbb{K}\)-linear map with \(k \geq n\)

\(\implies\) The following general “computation rule” holds:

For any \(x \in C\) and \(1 \leq j \leq n + 1\)

\[
\sum g(x_1 \otimes \cdots \otimes x_{j-1} \otimes f(x_j \otimes \cdots \otimes x_{j+n}) \otimes x_{j+n+1} \otimes \cdots \otimes x_{k+n+1})
= \sum g(x_1 \otimes \cdots \otimes x_{j-1} \otimes \bar{f}(x_j) \otimes x_{j+1} \otimes \cdots \otimes x_{k+1}).
\]
Proof of computation rule

Proof.

\[
\sum g(x_1 \otimes \cdots \otimes x_{j-1} \otimes f(x_j \otimes \cdots \otimes x_{j+n}) \otimes x_{j+n+1} \otimes \cdots \otimes x_{k+n+1})
\]

\[
= g \circ (\text{Id}^{\otimes j-1} \otimes f \otimes \text{Id}^{\otimes k-j+1}) \circ \Delta_{k+n}(x)
\]

\[
= g \circ (\text{Id}^{\otimes j-1} \otimes f \otimes \text{Id}^{\otimes k-j+1}) \circ (\text{Id}^{\otimes j-1} \otimes \Delta_n \otimes \text{Id}^{\otimes k-j+1}) \circ \Delta_k(x)
\]

\[
= g \circ (\text{Id}^{\otimes j-1} \otimes (f \circ \Delta_n) \otimes \text{Id}^{\otimes k-j+1}) \circ \Delta_k(x)
\]

\[
= g \circ (\text{Id}^{\otimes j-1} \otimes \overline{f} \otimes \text{Id}^{\otimes k-j+1}) \circ \Delta_k(x)
\]

\[
= \sum g(x_1 \otimes \cdots \otimes x_{j-1} \otimes \overline{f}(x_j) \otimes x_{j+1} \otimes \cdots \otimes x_{k+1}).
\]
Chapter 2.

Duality between Algebras and Coalgebras
Review: Some linear algebra (I)

\( \mathcal{V}, \mathcal{V}^* := \text{Hom}_\mathbb{K}(\mathcal{V}, \mathbb{K}) \) : a \( \mathbb{K} \)-vector space & its dual space

\( \langle \cdot, \cdot \rangle : \mathcal{V}^* \times \mathcal{V} \rightarrow \mathbb{K} \) : the natural pairing, i.e., \( \langle f, v \rangle := f(v) \)

If \( A \subset \mathcal{V} \) then \( A^\perp := \{ f \in \mathcal{V}^* | \langle f, v \rangle = 0, \forall v \in A \} \).

If \( B \subset \mathcal{V}^* \) then \( B^\perp := \{ v \in \mathcal{V} | \langle f, v \rangle = 0, \forall f \in B \} \).

\[ \implies \mathcal{V}^\perp = 0 \text{ and } \mathcal{V}^{\ast \perp} = 0. \]

\[ \implies \text{If } \varphi : \mathcal{V} \rightarrow \mathcal{W} \text{ is a } \mathbb{K} \text{-linear map of } \mathbb{K} \text{-vector spaces,} \]

then its \textit{transpose} \( \varphi^* : \mathcal{W}^* \rightarrow \mathcal{V}^* \) is uniquely defined by

\[ \langle \varphi^*(g), v \rangle = \langle g, \varphi(v) \rangle \quad \text{for all } g \in \mathcal{W}^* \text{ and } v \in \mathcal{V}. \]

(Note that it is just \( \varphi^* : \mathcal{W}^* \rightarrow \mathcal{V}^*, \ g \mapsto g \circ \varphi \).)
We define $\rho : V^* \otimes W^* \to (V \otimes W)^*$ by

$$\langle \rho(f \otimes g), v \otimes w \rangle := \langle f, v \rangle \langle g, w \rangle, \quad \forall f \in V^*, g \in W^*, v \in V, w \in W,$$

namely,

$$\rho(f \otimes g)(v \otimes w) := f(v)g(w).$$

Recall that the map $\rho$ is a canonical injection.

Moreover if one of $V$ and $W$ is finite dimensional, then the map $\rho$ becomes a $\mathbb{K}$-linear isomorphism.
The dual algebra of a coalgebra

Let \((C, \Delta, \epsilon)\) be a coalgebra over \(\mathbb{K}\) and \(C^* = \text{Hom}_\mathbb{K}(C, \mathbb{K})\) be its dual space.

We can define \(M : C^* \otimes C^* \to C^*\) and \(u : \mathbb{K} \to C^*\) by

\[
M : C^* \otimes C^* \xrightarrow{\rho} (C \otimes C)^* \xrightarrow{\Delta^*} C^* \quad \text{and} \quad u : \mathbb{K} \xrightarrow{\sim} \mathbb{K}^* \xrightarrow{\epsilon^*} C^*.
\]

Proposition

1. \((C^*, M, u)\) is an algebra over \(\mathbb{K}\).

2. If \(g : C \to D\) is a morphism of coalgebras then \(g^* : D^* \to C^*\) is a morphism of algebras.
The dual coalgebra of a finite dimensional algebra

Let $(A, M, u)$ be a finite dimensional algebra over $\mathbb{K}$ and $A^* = \text{Hom}_\mathbb{K}(A, \mathbb{K})$ be its dual space.

In this case, the map $\rho : A^* \otimes A^* \rightarrow (A \otimes A)^*$ is bijective.

Thus we can define $\Delta : A^* \rightarrow A^* \otimes A^*$ and $\epsilon : A^* \rightarrow \mathbb{K}$ by

$\Delta : A^* \xrightarrow{M^*} (A \otimes A)^* \xrightarrow{\rho^{-1}} A^* \otimes A^*$ and

$\epsilon : A^* \xrightarrow{u^*} \mathbb{K}^* \xrightarrow{\sim} \mathbb{K}$.

Proposition

1. $(A^*, \Delta, \epsilon)$ is a coalgebra over $\mathbb{K}$.

2. If $f : A \rightarrow B$ is a morphism of algebras then $f^* : B^* \rightarrow A^*$ is a morphism of coalgebras.
Categorical duality for finite dimensional case

\((A, M, u)\): a finite dimensional algebra

\((C, \Delta, \epsilon)\): a finite dimensional coalgebra

If \(V\) is a finite dimensional vector space, then recall that

\[\mathcal{E} : V \to V^{**}, \mathcal{E}(v)(f) := f(v), \quad \forall v \in V, f \in V^{*}\]

is an isomorphism.

Proposition

1. \(\mathcal{E} : A \to A^{**}\) is an isomorphism of algebras;

2. \(\mathcal{E} : C \to C^{**}\) is an isomorphism of coalgebras.

\[\Rightarrow\text{ The category } F \text{ Coalg is anti-equivalent to the category } F \text{ Alg.}\]

Also, we have \(F \text{ Cocomm.Coalg} \overset{\sim}{\leftrightarrow} F \text{ Comm.Alg.}\)
Sub-coalgebras of a coalgebra & its duality

Let \((C, \Delta, \epsilon)\) be a coalgebra.

If \(V\) is a subspace of \(C\) that satisfies \(\Delta(V) \subseteq V \otimes V\), then clearly \((V, \Delta|_V, \epsilon|_V)\) becomes a coalgebra and it is easy to check that the inclusion map \(V \hookrightarrow C\) is a morphism of coalgebras.

This fact naturally leads to the following definition:

Definition

A subspace \(V \subseteq C\) is called a sub-coalgebra if \(\Delta(V) \subseteq V \otimes V\).

Proposition

1. If \(V \subseteq C\) is a sub-coalgebra, \(V^\perp\) is a (two-sided) ideal of \(C^*\).

2. If \(J \subseteq C^*\) is a (two-sided) ideal, \(J^\perp\) is a sub-coalgebra of \(C\).
Coideals of a coalgebra & its duality

Let \((C, \Delta, \epsilon)\) be a coalgebra.

**Definition**

A subspace \(V \subseteq C\) is called a (two-sided) **coideal** if

1. \(\Delta(V) \subseteq V \otimes C + C \otimes V\);
2. \(\epsilon(V) = 0\).

**Proposition**

1. If \(V \subseteq C\) is a coideal, \(V^\perp\) is a subalgebra of \(C^*\).
2. If \(B \subseteq C^*\) is a subalgebra, \(B^\perp\) is a coideal of \(C\).
Kernel and image for a morphism of coalgebras

Let $g : C \to D$ be a morphism of coalgebras.

**Proposition**

1. $\ker g$ is a coideal in $C$;
2. $\text{Im } g$ is a co-subalgebra in $D$.

If $J \subseteq C$ is a coideal, there is a unique coalgebra structure on $C/J$ such that $\pi : C \to C/J$ is a morphism of coalgebras.

**Homomorphism Theorem**

If $J \subseteq \ker g$ is a coideal, there is a unique morphism of coalgebras $\bar{g} : C/J \to D$ such that $\bar{g} \circ \pi = g$. In particular, $C/\ker g \cong \text{Im } g$. 
Left and right coideals of a coalgebra & its duality

Let $(C, \Delta, \epsilon)$ be a coalgebra.

Definition

1. A subspace $V \subseteq C$ is called a left coideal if $\Delta(V) \subseteq C \otimes V$;
2. A subspace $V \subseteq C$ is called a right coideal if $\Delta(V) \subseteq V \otimes C$.

Proposition

1. If $V \subseteq C$ is a left (right) coideal, $V^\perp$ is a left (right) ideal in $C^*$;
2. If $J \subseteq C^*$ is a left (right) ideal, $J^\perp$ is a left (right) coideal in $C$. 
Caution!!

A coideal need not be either a left or a right coideal.

Furthermore, if $V \subseteq C$ is both a left and right coideal, then $V$ is a sub-coalgebra and not a coideal unless $V = 0$.

This is because $(V \otimes C) \cap (C \otimes V) = V \otimes V$.

(Or, simply, by duality.)
Chapter 3.
Bialgebras and Hopf Algebras
The tensor product of two coalgebras is a coalgebra.

\[(C, \Delta_C, \epsilon_C), (D, \Delta_D, \epsilon_D) : \text{coalgebras over } K\]

\[T : C \otimes D \to D \otimes C : \text{the ‘twist’ map, i.e., } c \otimes d \mapsto d \otimes c\]

We can define \(\Delta_{C \otimes D}\) by

\[
\Delta_{C \otimes D} : C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\text{Id} \otimes T \otimes \text{Id}} C \otimes D \otimes C \otimes D.
\]

Also, we can define \(\epsilon_{C \otimes D}\) by

\[
\epsilon_{C \otimes D} : C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} K \otimes K \xrightarrow{\sim} K.
\]

Proposition

\((C \otimes D, \Delta_{C \otimes D}, \epsilon_{C \otimes D})\) is a coalgebra.
Definition of bialgebras (bigebras)

Suppose there is a system \((H, M, u, \Delta, \epsilon)\) such that
\((H, M, u)\) is an algebra and \((H, \Delta, \epsilon)\) is a coalgebra.

Proposition

The following are equivalent:

(A). \(M : H \otimes H \rightarrow H\) and \(u : \mathbb{K} \rightarrow H\) are coalgebra morphisms;

(B). \(\Delta : H \rightarrow H \otimes H\) and \(\epsilon : H \rightarrow \mathbb{K}\) are algebra morphisms;

Definition

\((H, M, u, \Delta, \epsilon)\) is called a **bialgebra** if one of (A) and (B) holds.
‘Convolution product’ $\ast$ in $\text{Hom}_K(C, A)$

$(A, M, u) :$ an algebra over $K$

$(C, \Delta, \epsilon) :$ a coalgebra over $K$

$H := \text{Hom}_K(C, A):$ the set of all $K$-linear maps from $C$ to $A$

We define so called the ‘convolution product’ $\ast : H \otimes H \to H$ by

$\ast : H \otimes H \leftrightarrow \text{Hom}_K(C \otimes C, A \otimes A) \xrightarrow{\text{Hom}(\Delta, M)} H,$

where the first map is a canonical injection, and

the second map $\text{Hom}(\Delta, M)$ is the composition map defined by

$\text{Hom}(\Delta, M) : \varphi \mapsto M \circ \varphi \circ \Delta.$
Unit of $\text{Hom}_K(C, A)$ with respect to convolution product

Similarly, $\epsilon : C \to K$ and $u : K \to A$ induce $\eta : K \to H$ defined by

$$\eta : K \cong \text{Hom}_K(K, K) \xrightarrow{\text{Hom}(\epsilon, u)} H = \text{Hom}_K(C, A),$$

where $\text{Hom}(\epsilon, u) : \varphi \mapsto u \circ \varphi \circ \epsilon$.

Consequently, we obtain the following result:

**Proposition**

1. $(\text{Hom}_K(C, A), *, \eta)$ is an algebra over $K$;
2. The identity element in $\text{Hom}_K(C, A)$ is $\eta(1_K) = u \circ \epsilon$. 
Definition of Hopf algebras

\((H, M, u, \Delta, \epsilon) : \text{a bialgebra over } K\)

Put \(H^A := (H, M, u)\) and \(H^C := (H, \Delta, \epsilon)\).

Definition

\((H, M, u, \Delta, \epsilon)\) is a Hopf algebra if \(\text{Id} : H \to H\) has inverse \(S : H \to H\)
in the algebra \((\text{Hom}_K(H^C, H^A), *, \eta)\). \(S\) is called the antipode.

In other words, there is \(S : H \to H\) commuting the following diagram:
Ex. 1. ‘Group algebra’

Let $G$ be a group and $\mathbb{K}G$ be a group algebra over $\mathbb{K}$. $\mathbb{K}G$ is a bialgebra if we endow $\mathbb{K}G$ with 'group-like coalgebra'. $\mathbb{K}G$ is a Hopf algebra with $S : \mathbb{K}G \to \mathbb{K}G$, $g \mapsto g^{-1}$, $\forall g \in G$. It is cocommutative, and it is commutative iff $G$ is abelian.

Ex. 2. ‘The set $\mathbb{K}^G$ of all functions from a finite group $G$ to $\mathbb{K}$’

$\mathbb{K}^G$ is an algebra with pointwise addition and multiplication and a coalgebra with $\Delta(\varphi)(g, h) := \varphi(gh)$ and $\epsilon(\varphi) := \varphi(1_G)$. $\mathbb{K}^G$ is a Hopf algebra with $S(\varphi)(g) := \varphi(g^{-1})$. It is commutative, and it is cocommutative iff $G$ is abelian.
Examples of Hopf algebras (II)

Ex. 3. ‘Tensor algebra’ & its families

Let $T(V) = \bigoplus_{j=0}^{\infty} V \otimes j$ be a tensor algebra over a $K$-space $V$. If, for all $v \in V$, we define $\Delta(v) := 1 \otimes v + v \otimes 1$, $\epsilon(v) := 0$, and $S(v) := -v$, then $T(V)$ is a cocomutative Hopf algebra.

‘Symmetric algebra’ and ‘Exterior algebra’ are Hopf algebras.

Ex. 4. ‘Universal enveloping algebra of a Lie algebra’

Let $U(\mathfrak{g})$ be a U.E.A. of a Lie algebra $\mathfrak{g}$ over $K$. If, for all $X \in \mathfrak{g}$, we define $\Delta(X) := 1 \otimes X + X \otimes 1$, $\epsilon(X) := 0$, and $S(X) := -X$, then $U(\mathfrak{g})$ is a cocomutative Hopf algebra.

It is commutative if and only if $\mathfrak{g}$ is abelian.
Ex. 5. ‘Sweedler’s 4-dimensional Hopf algebra’

Assume that $\text{char } \mathbb{K} \neq 2$.

If $H$ is generated as an algebra by $c$ and $x$ by the relations

$$c^2 = 1, \quad x^2 = 0, \quad xc = -cx,$$

then $H$ is a 4-dimensional $\mathbb{K}$-space with basis $\{1, c, x, cx\}$.

The coalgebra structure of $H$ is defined by

$$\Delta(c) := c \otimes c, \quad \Delta(x) := c \otimes x + x \otimes 1, \quad \epsilon(c) := 1, \quad \epsilon(x) := 0.$$

If $S(c) := c^{-1}$, $S(x) := -cx$, then $H$ is a Hopf algebra.

This is the smallest example which is both non-commutative and non-cocommutative.
Chapter 4.

Duality between

Linear Algebraic Groups and Hopf Algebras
From now on,

we suppose that $\mathbb{K}$ is algebraically closed.
Linear algebraic groups (=Affine algebraic groups)

Definition

An algebraic group $G$ is an algebraic variety (over $\mathbb{K}$) which is also a group such that the maps defining the group structure $\mu : G \times G \to G, (g, h) \mapsto gh$ and $\iota : G \to G, g \mapsto g^{-1}$ are morphisms of varieties. (Here, $G \times G$ is the product of varieties.)

Definition

An algebraic group is called linear if the underlying variety is affine.

Definition

A homomorphism $G \to G'$ of algebraic groups is defined as a variety morphism which is also a group homomorphism.
In algebraic geometry, there is a well-known (anti-)correspondence between algebra and geometry via Nullstellensatz.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Affine variety $V$</td>
<td>Affine algebra $\mathbb{K}[V]$</td>
</tr>
<tr>
<td>Points in $V$</td>
<td>Maximal ideals in $\mathbb{K}[V]$</td>
</tr>
<tr>
<td>Irr. closed sub-varieties of $V$</td>
<td>Prime ideals in $\mathbb{K}[V]$</td>
</tr>
<tr>
<td>Variety morphism $V_1 \rightarrow V_2$</td>
<td>Algebra morphism $\mathbb{K}[V_2] \rightarrow \mathbb{K}[V_1]$</td>
</tr>
<tr>
<td>(Categorical) Product $V_1 \times V_2$</td>
<td>Coproduct $\mathbb{K}[V_1] \otimes\mathbb{K}[V_2]$</td>
</tr>
<tr>
<td>Combinatorial dimension</td>
<td>Krull Dimension</td>
</tr>
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</tbody>
</table>
Duality between linear algebraic groups & Hopf algebras

<table>
<thead>
<tr>
<th>Linear algebraic groups $G$</th>
<th>$\iff$</th>
<th>(comm.) Hopf algebra $\mathbb{K}[G]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Affine variety $G$</td>
<td>$\iff$</td>
<td>Affine algebra $\mathbb{K}[G]$</td>
</tr>
<tr>
<td>Map $G_1 \to G_2$</td>
<td>$\iff$</td>
<td>Map $\mathbb{K}[G_2] \to \mathbb{K}[G_1]$</td>
</tr>
<tr>
<td>(Categorical) Product $G \times G$</td>
<td>$\iff$</td>
<td>Coproduct $\mathbb{K}[G] \otimes \mathbb{K}[G]$</td>
</tr>
<tr>
<td>$\mu : G \times G \to G$</td>
<td>$\iff$</td>
<td>$\mu^0 = \Delta : \mathbb{K}[G] \to \mathbb{K}[G] \otimes \mathbb{K}[G]$</td>
</tr>
<tr>
<td>$\iota : G \to G$</td>
<td>$\iff$</td>
<td>$\iota^0 = S : \mathbb{K}[G] \to \mathbb{K}[G]$</td>
</tr>
<tr>
<td>Associativity of $\mu$</td>
<td>$\iff$</td>
<td>Coassociativity of $\Delta$</td>
</tr>
<tr>
<td>Existence of identity</td>
<td>$\iff$</td>
<td>Defining property of counit</td>
</tr>
<tr>
<td>Existence of inverse</td>
<td>$\iff$</td>
<td>Defining property of antipode</td>
</tr>
</tbody>
</table>

For $(\mathbb{K}[G], M, u, \Delta, \epsilon, S)$, $M(\varphi, \psi)(g) = \varphi(g)\psi(g)$, $u(1\mathbb{K}) = 1\mathbb{K}$, $\Delta(\varphi)(g, h) = \varphi(gh)$, $\epsilon(\varphi) = \varphi(1_G)$, and $S(\varphi)(g) = \varphi(g^{-1})$. 
Put $A := \mathbb{K}[G]$ and $M^0 = \text{diag} : G \to G, g \mapsto (g, g)$. 

\[
\begin{align*}
G \times G \times G & \xrightarrow{\text{Id} \times \mu} G \times G \\
\mu \times \text{Id} & \downarrow \downarrow \mu \\
G \times G & \xrightarrow{\mu} G
\end{align*}
\]

$A \otimes A \otimes A \xleftarrow{\text{Ax.1}} A \otimes A$

$\Delta \otimes \text{Id} \uparrow \uparrow \Delta$

$A \otimes A \leftarrow \Delta$

$A \otimes A$

\[
\begin{align*}
G & \xrightarrow{\text{Id}} G \\
\text{Id} & \downarrow \downarrow g \mapsto (1_G, g) \\
G & \xrightarrow{\mu} G \times G
\end{align*}
\]

$A \xleftarrow{\text{Ax.2}} \mathbb{K} \otimes A$

$\approx \uparrow \uparrow \epsilon \otimes \text{Id}$

$A \otimes \mathbb{K} \leftarrow \text{Id} \otimes \epsilon$

$A \otimes A$

\[
\begin{align*}
G \times G & \xrightarrow{\iota \times \text{Id}} G \times G \\
\text{diag} & \xrightarrow{g \mapsto 1_{\mathbb{K}}} \mathbb{K} \xrightarrow{1_{\mathbb{K}} \mapsto 1_G} G \\
\text{diag} & \xrightarrow{\mu} G \times G
\end{align*}
\]

$A \otimes A \xleftarrow{\text{Ax.3}} A \otimes A$

$\Delta \xrightarrow{\Delta} \mathbb{K} \xleftarrow{\epsilon} A$

$A \otimes A \leftarrow \text{Id} \otimes S$

$A \otimes A$
Final comment:

The study of ‘Quantum groups’ (they are some kind of Hopf algebras) is a study for deformation of this duality between linear algebraic groups and Hopf algebras.
Thank you for your attention!

Enjoy Hopf algebra theory!!