Combinatorics in Hungary and Extremal Set Theory

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Colloquium Talk

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Combinatorics in Hungary

A little history.
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Revolution and freedom fight against **Austria** in 1848, one and a half year long war. Hungary was winning, **Austria** asked the help of the **Russians**. The two big countries easily suppressed the revolution.

Agreement in 1867. The Austrian Monarchy became the **Austro-Hungarian Monarchy**. Very fast economic progress in the Hungarian part.
Combinatorics in Hungary

Educational reforms on every level.
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General level of elementary schools raised. Talent search. Good math professor at universities. One of them: Gyula König.
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The first mathematical contest for high school students in the world in 1920.
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The world very first mathematical journal for high school students: 1894!

The first mathematical contest for high school students in the world in 1894!

Harsányi (NP), Von Neumann, Teller, Wigner (NP) came from the same high school in Budapest. Szilárd went to another strong school.
Marriage problem = Perfect Matching

Given $n$ girls and $n$ boys, their “knowing each other” is given with a bipartite graph.
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Can all the girls find husbands?
Marriage problem = Perfect Matching

Given $n$ girls and $n$ boys, their “knowing each other” is given with a bipartite graph.

Necessary condition: every set $A$ of girls need to know at least $|A|$ boys.
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Given $n$ girls and $n$ boys, their “knowing each other” is given with a bipartite graph.

Theorem (Dénes König, 1916, Hall, 1935) The condition is necessary and sufficient.
First book in Graph Theory by Dénes König, 1936.
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Under his influence Erdős, Gallai, Szekeres, Turán started to think about graphs.
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Erdős went to England and the USA.

Turán was in a labor camp, yet he was doing graph theory there!
Notation: $[n] = \{1, 2, \ldots, n\}$.

A family $\mathcal{F} \subseteq 2^{[n]}$ is **intersecting** if $F \cap G \neq \emptyset$ holds for every pair $F, G \in \mathcal{F}$. 
Notation: $[n] = \{1, 2, \ldots, n\}$.

A family $\mathcal{F} \subset 2^{[n]}$ is **intersecting** if $F \cap G \neq \emptyset$ holds for every pair $F, G \in \mathcal{F}$. 
Erdős, Ko, Rado

**Theorem** (Erdős – Ko – Rado, done in 1938, published in 1961) If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting where $k \leq \frac{n}{2}$ then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$
I found a short proof in 1972. Erdős said: it is from the **BOOK**.
Erdős, Ko, Rado

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Erdős returned to Hungary in 1954.
Alfréd Rényi a strong probabilist, director of the Mathematical Institute of the Hungarian Academy of Sciences.
Rényi Alfréd (1921-1970)
Random graphs

Alfréd Rényi a strong probabilist, director of the Mathematical Institute of the Hungarian Academy of Sciences.

Erdős and Rényi started to study RANDOM GRAPHS.
Random graphs

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3 different models:

(1) One graph is chosen randomly from the possible

$$\binom{n}{2}^e$$

graphs with \( e \) edges.
(2) Choose the edges independently with probability

\[ p = \frac{e}{\binom{n}{2}}. \]
Random graphs

(3) Add edges one by one and look at the graph when $e$ edges are added. They are equivalent for reasonable problems.
Random graphs

**Theorem** The first cycle appears around $e = \frac{n}{2}$.

**Theorem** The graph becomes connected around $e = \frac{1}{2}n \log n$. 
Random graphs

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**Theorem** The graph becomes connected around $e = \frac{1}{2}n \log n$.

Before that, “giant component”.
Random graphs

Many applications in natural sciences.

E.g. physics, social graphs, of course, with other probabilities.
Shadows

**Definition.** The **shadow** $\sigma(\mathcal{A})$ of $\mathcal{A}$ is the family of all $k-1$-element sets obtained from the members of $\mathcal{A}$ by deleting exactly one element.

$$\sigma(\mathcal{A}) = \{ B : |B| = k - 1, B \subset A \in \mathcal{A} \}$$
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Shadows

Given $n, k$ and $|\mathcal{F}|$, minimize $|\sigma(\mathcal{F})|$. If lucky then $|\mathcal{F}| = \binom{a}{k}$ holds for an integer $a$ then the best construction is

$$\min |\sigma(\mathcal{F})| = \binom{a}{k-1}$$
Lemma If $0 < k, m$ are integers then one can find integers $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$ such that

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$$

and they are unique.

This is called the **canonical form** of $m$. 
otherwise

Shadow Theorem (Kruskal-K, 1960’s) If \( n, k \) and \(|\mathcal{F}|\) are given the canonical form of \(|\mathcal{F}|\) is

\[
|\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k - 1} + \ldots + \binom{a_t}{t}
\]

then

\[
\min |\sigma(\mathcal{F})| = \binom{a_k}{k - 1} + \binom{a_{k-1}}{k - 2} + \ldots + \binom{a_t}{t - 1}.
\]
Szemerédi’s regularity lemma
Szemerédi’s regularity lemma

Large graphs follow some pattern, some rules.
Szemerédi’s regularity lemma

edge density = \frac{\text{number of edges}}{N^2}
Szemerédi’s regularity lemma

(edge density = \( \frac{\text{number of edges}}{uv} \))
Szemerédi’s regularity lemma

This graph is $\varepsilon$-regular (randomlike) if for every $\varepsilon > 0$, $|U|, |V| > \varepsilon N$

\[ |\text{edge density} - \text{edge density}| < \varepsilon \]
Szemerédi’s regularity lemma

Regularity lemma (Szemerédi, 1975)

A graph with many (many-many) vertices can be partitioned into equally sized subsets in such a way that most pairs of subsets span an $\varepsilon$-regular bipartite graph.
A problem motivated by cryptology

\[ A, B \subset \binom{[n]}{3}, |A| = |B| \]

\[ \sigma(A) \cap \sigma(B) = \emptyset \]

(In other words, if \( A \in \mathcal{A}, B \in \mathcal{B} \) then \( |A \cap B| \leq 1 \).)

Find

\[ f(n, 3) = \max |A| \text{ under these conditions.} \]
Only an estimate

If $x$ is a real number, \[ \binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}. \]

**Theorem.** (Lovász’ version of the Shadow theorem) If $\mathcal{A}$ is a family of $k$-element sets,

\[ |\mathcal{A}| = \binom{x}{k}, \]

then

\[ |\sigma(\mathcal{A})| \geq \binom{x}{k}. \]
A weak estimate from **Shadow Theorem**

Choose $x$ in this way: $|A| = |B| = \binom{x}{3}$

By the Shadow Theorem: $\binom{x}{2} \leq |\sigma(A)|, |\sigma(B)|$

$$2 \binom{x}{2} \leq |\sigma(A)| + |\sigma(B)| \leq \binom{n}{2}$$

From here, asymptotically

$$|A| = \binom{x}{3} \leq \frac{1}{2\sqrt{2}} \binom{n}{3} (1 + o(1))$$
Trivial construction gives:

\[ |\mathcal{A}| = \frac{n^3}{48} (1 + o(1)). \]

\[ \mathcal{A} = \left\{ (a, b, c) : a < b < c \leq \frac{n}{2} \right\}, \quad \mathcal{B} = \left\{ (a, b, c) : \frac{n}{2} \leq a < b < c \right\}. \]
A better construction: $|\mathcal{A}| = \frac{n^3}{24} (1 + o(1))$.

$\mathcal{A} = \{(a, b, c) : \frac{b+c}{2} \leq \frac{n}{2}\}$

$\{B = \{(a, b, c) : \frac{n}{2} < \frac{a+b}{2}\}.$
We have $0.25 \leq \limsup \frac{f(n,3)}{\binom{n}{3}} \leq 0.35355...$.
We have $0.25 \leq \lim \sup \frac{f(n, 3)}{\binom{n}{3}} \leq 0.35355\ldots$.

**Theorem (Frankl-Kato-Katona-Tokushige)**

\[ f(n, 3) = 0.278\ldots \binom{n}{3}(1 + o(1)) \]
Theorem (Frankl-Kato-Katona-Tokushige)

\[ f(n, 3) = \kappa^3 \binom{n}{3} (1 + o(1)) \]

where \( \kappa \) is the unique real root in the (0,1)-interval of the equation

\[ z^3 = (1 - z)^3 + 3z(1 - z)^2. \]
Theorem (Frankl-Kato-Katona-Tokushige, 2013)

\( \mathcal{A}_1, \mathcal{A}_2 \subset \binom{\left[ n \right]}{k}, |\mathcal{A}_1| = |\mathcal{A}_2|, \) and

\( \mathcal{A}_1 \in \mathcal{A}_1, \mathcal{A}_2 \in \mathcal{A}_2 \) imply \( |\mathcal{A}_1 \cap \mathcal{A}_2| \leq 1 \) then

\[
\max |\mathcal{A}| = \mu_k^k \binom{n}{k} (1 + o(1))
\]

where \( \mu_k \) is the unique real root in the \((0,1)\)-interval of the equation

\[
z^k = (1 - z)^k + k z(1 - z)^{k-1}.
\]
Some months later Huang, Linial, Naves, Peled, Sudakov proved a very similar result. They considered two families $A \subset \binom{[n]}{k}$, $B \subset \binom{[n]}{\ell}$ where $A \in \mathcal{A}, b \in B$ implies $|A \cap B| \leq 1$. Supposing

$$\frac{|A|}{\binom{n}{k}} = \alpha$$

they asymptotically determine

$$\max \frac{|B|}{\binom{n}{\ell}}.$$ 

The case $k = \ell$ gives back our result, but their upper estimate is less sharp.
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The case $k = \ell$ gives back our result, but their upper estimate is less sharp.

Their motivation is theoretical.
Badly needed generalizations

$s$-shadows $(s \neq k - 2)$, that is, $|A \cap B| \leq r$. 
Badly needed generalizations

*s-shadow* $(s \neq k - 2)$, that is, $|A \cap B| \leq r$.

More families rather than only 2.
A family $\mathcal{F} \subset 2^{[n]}$ is called a $(u, v)$-union-intersecting if for different members $F_1, \ldots, F_u, G_1, \ldots, G_v$ the following holds:

$$(\bigcup_{i=1}^{u} F_i) \cap (\bigcup_{j=1}^{v} G_j) \neq \emptyset.$$
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**Theorem (Katona-D.T. Nagy 2014+)** Let $1 \leq u \leq v$ and suppose that the family $\mathcal{F} \subset {[n] \choose k}$ is a $(u,v)$-union–intersecting family then

$$|\mathcal{F}| \leq \binom{n-1}{k-1} + u - 1$$

holds if $n > n(k,v)$. 
A family $\mathcal{F} \subset 2^{[n]}$ is called a $(u,v)$-union-intersecting if for different members $F_1, \ldots, F_u, G_1, \ldots, G_v$ the following holds:

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**Theorem (Katona-D.T. Nagy 2014+)** Let $1 \leq u \leq v$ and suppose that the family $\mathcal{F} \subset \left(\begin{array}{c} [n] \\ k \end{array}\right)$ is a $(u,v)$-union–intersecting family then

$$|\mathcal{F}| \leq \binom{n-1}{k-1} + u - 1$$

holds if $n > n(k, v)$. 
谢谢
不耻下问