Positive semidefinite minimum rank of a sign pattern matrix

Xinmao Wang

School of Mathematical Sciences
University of Science and Technology of China

April 5, 2014
@ Shanghai Jiao Tong University
Definitions

- A **sign pattern matrix** is a matrix whose entries come from \{+, −, 0\} or \{1, −1, 0\}.

- \(\text{sgn}(M)\) is the sign pattern matrix whose entries are the signs of the corresponding entries in a real matrix \(M\).

- The **minimum rank** and **positive semidefinite minimum rank** of a sign pattern matrix \(A\) are
  \[
  \text{mr}(A) = \min_{\text{sgn}(M)=A} \text{rank}(M) \\
  \text{pmr}(A) = \min_{\text{sgn}(M)=A \text{ and } M \succeq 0} \text{rank}(M)
  \]

- \(G(A)\) is the simple graph whose adjacency matrix is corresponding to the sign pattern matrix \(A\).
Facts

- Obviously, \( mr(A) \leq pmr(A) \) for any symmetric sign pattern matrix \( A \) with positive diagonal entries.

- The following example shows that the gap between \( mr(A) \) and \( pmr(A) \) can be very large.

- Example.

\[
A = \begin{pmatrix}
1 & -1 & \cdots & -1 \\
-1 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & -1 \\
-1 & \cdots & -1 & 1
\end{pmatrix}_{n \times n}, \quad n \geq 5.
\]

- \( mr(A) \leq \frac{n+1}{2}, \quad pmr(A) = n - 1. \)
Main Problem

- Given an $n \times n$ symmetric sign pattern matrix $A$ whose diagonal entries are all positive. Determine $\text{pmr}(A)$. 
Bounds via the number of signs

**Lemma.** Let $\alpha_1, \cdots, \alpha_n$ be vectors, $r = \dim \langle \alpha_1, \cdots, \alpha_n \rangle$.

- If $\alpha_j \cdot \alpha_j < 0, \forall i \neq j$, then $n \leq r + 1$.
- If $\alpha_j \cdot \alpha_j \leq 0, \forall i \neq j$, then $n \leq 2r$.

**Lemma.** Let $x_1, \cdots, x_k \in \mathbb{N}$ such that $x_1 + \cdots + x_k = n$. If and only if $x_i$ are almost equal, $x_1^2 + \cdots + x_k^2$ reaches the minimum.

**Write** $f(k, n) = \min_{x_1 + \cdots + x_k = n, x_i \in \mathbb{N}} x_1^2 + \cdots + x_k^2$.

**Lemma.** $f(k, n) \geq \frac{n^2}{k}$. 
Bounds via the number of signs

- **Theorem.** Let $M$ be an $n \times n$ positive semidefinite real matrix with $\text{rank}(M) = r$, $N_+, N_-, N_0$ be the number of $+$, $-$, 0 entries of $M$, respectively.
  - $N_+ \geq f(2r, n)$;
  - $N_- \leq n^2 - f(r + 1, n)$;
  - $N_0 \leq n^2 - f(r, n)$.

- **Corollary.**
  \[
  r \geq \max \left( \frac{n^2}{2N_+}, \frac{N_-}{n^2 - N_-}, \frac{n^2}{n^2 - N_0} \right).
  \]
Bounds via the number of signs

❖ The above bounds is very rough, since two matrices with the same number of signs may have different ranks.

❖ Example.

\[ A_1 = \begin{pmatrix} I_{k^2-k} & O \\ O & J_{k^2+k} \end{pmatrix}, \quad A_2 = \begin{pmatrix} I_{k^2} & J \\ J & I_{k^2} \end{pmatrix}, \]

where \( I \) is the identity matrix, \( J \) is the matrix of all ones.

❖ \( A_1 \) and \( A_2 \) have the same number of signs;

❖ \( \text{pmr}(A_1) = 2, \text{pmr}(A_2) = 2k^2 - 1. \)
Bounds via pmr of submatrices

- For any positive semidefinite matrix $M = \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{pmatrix}$,
  
  $\text{rank}(M) \leq \text{rank}(M_1) + \text{rank}(M_3)$.

- However, for a signed pattern matrix $A = \begin{pmatrix} A_1 \\ A_2^T \\ A_2 \\ A_3 \end{pmatrix}$,
  
  $\text{pmr}(A) \leq \text{pmr}(A_1) + \text{pmr}(A_3)$

  does not always holds.

  For example,
  
  $\text{pmr}\begin{pmatrix} J & I_k \\ I_k & J \end{pmatrix} = k + 1$. 

Bounds via pmr of submatrices

- **Theorem.** Let \( A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix} \) where \( A_1 : n_1 \times n_1 \).

  - \( \text{pmr}(A) \geq \text{pmr}(A_1) \).
  - \( \text{pmr}(A) \leq \text{pmr}(A_1) + n - n_1 \) when \( N_0(A_1) = 0 \).
  - \( \text{pmr}(A) \leq n - 1 \) when \( A \neq I \).
  - Suppose \( N_0(A) = 0 \). \( \text{pmr}(A) = n - 1 \) if and only if \( A = P(2I - J)P^T \) for a signed permutation matrix \( P \).
Bounds via pmr of submatrices

- If $A_1 = P \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{pmatrix} P^T$ where $P$ is a signed permutation matrix and $N_0(A_{ii}) = 0$, $\forall i$, then $\text{pmr}(A) \leq \text{pmr}(A_1) + k(n - n_1)$.

Remark.

- The minimal possible $k$ above is the chromatic number of the complement graph of $G(A_1)$.
- The above inequality is tight.
Bounds via pmr of submatrices

Example.

\[
A = \begin{pmatrix}
\text{diag}(B_1, \ldots, B_k) & J \\
J & 1
\end{pmatrix}
\]

where each \( B_i = 2I_r - J \).

- \( n_1 = kr, n = n_1 + 1 \).
- \( \text{pmr}(A_1) = kr - k \).
- \( \text{pmr}(A) = n - 1 \).
- \( \text{pmr}(A) = \text{pmr}(A_1) + k(n - n_1) \).
Bounds via graph structure

- **Theorem.** Suppose

\[ A = \begin{pmatrix} A_{11} & O & A_{13} \\ O & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix}, \quad B_i = \begin{pmatrix} A_{ii} & A_{i3} \\ A_{i3}^T & A_{33} \end{pmatrix}, \]

where \( A_{33} : k \times k \), then

\[ 0 \leq \text{pmr}(B_1) + \text{pmr}(B_2) - \text{pmr}(A) \leq 2(k - 1). \]

- The above theorem can also be stated as follows.

- Suppose \( G(A) = G(B_1) \cup G(B_2) \) is the union of two induced subgraphs, \( k \) is the number of vertices in \( G(B_1) \cap G(B_2) \), then

\[ 0 \leq \text{pmr}(B_1) + \text{pmr}(B_2) - \text{pmr}(A) \leq 2(k - 1). \]
Bounds via graph structure

❖ Corollary.

➢ If \( k = 1 \), then \( \text{pmr}(A) = \text{pmr}(B_1) + \text{pmr}(B_2) \).

➢ If \( G(A) \) is a tree, then \( \text{pmr}(A) = n - 1 \).

➢ If \( G(A) \) is unicyclic, then \( \text{pmr}(A) = n - 1 \) or \( n - 2 \).

\[
\text{pmr}
\begin{pmatrix}
1 & -1 & \delta \\
-1 & 1 & \vdots \\
\vdots & \vdots & -1 \\
\delta & -1 & 1
\end{pmatrix}
= \begin{cases}
    n - 1, & \delta = -1; \\
    n - 2, & \delta = +1.
\end{cases}
\]

❖ Question. \( \text{pmr}(B_1) + \text{pmr}(B_2) - \text{pmr}(A) \leq k - 1? \)
Reference

- Arav, Marina; Hall, Frank J.; Li, Zhongshan; van der Holst, Hein; Zhang, Lihua; Zhou, Wenyan; The minimum rank of a sign pattern matrix with a 1-separation. *Linear Algebra Appl.* 448 (2014), 205–216.
- Li, Zhongshan; Gao, Yubin; Arav, Marina; Gong, Fei; Gao, Wei; Hall, Frank J.; van der Holst, Hein; Sign patterns with minimum rank 2 and upper bounds on minimum ranks. *Linear Multilinear Algebra* 61 (2013), no. 7, 895–908.


Thank you!