Perfect sets and \( f \)-ideals

Author Jin Guo

(This is a joint work with professor Tongsuo Wu)

Department of Mathematics, Shanghai Jiaotong University

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Perfect sets and $f$-ideals

Jin Guo

Outline

1 Introduction

2 Perfect sets and $f$-ideals of degree $d$

3 $(n, 2)^{th}$ perfect number

4 Structure of $V(n, 2)$

5 Further works

6 References
A bridge between algebra and combinatorics

Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References
A bridge between algebra and combinatorics

Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References

From simplicial complex to ideals

Simplicial complex $\Delta \longrightarrow$

1. Stanley-Reisner ideal $I_\Delta$; 2. Facet ideal $I(\Delta)$. 
A bridge between algebra and combinatorics

**From simplicial complex to ideals**

Simplicial complex $\Delta \rightarrow$
1. Stanley-Reisner ideal $I_\Delta$; 2. Facet ideal $I(\Delta)$.

**From ideal to simplicial complexes**

Square-free monomial ideal $I \rightarrow$
1. Stanley-Reisner complex $\delta_N(I)$; 2. Facet complex $\delta_F(I)$. 
Two simplicial complexes

Perfect sets and f-ideals

Jin Guo

Outline

Introduction

Perfect sets and f-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References
Two simplicial complexes

Simplicial complex

A simplicial complex $\Delta$ on $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$, satisfying:

- $\{i\} \in \Delta$ for all $i \in [n]$;
- If $F \in \Delta$, and $G \subseteq F$, then $G \in \Delta$ (including $\emptyset$).
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**Simplicial complex**

- **Face**: element of $\Delta$;
Two simplicial complexes

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- **Face**: element of \( \Delta \);
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Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals

of degree $d$

$(n, 2)^{th}$

perfect number

Structure of

$V(n, 2)$

Further works

References

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Two simplicial complexes

Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References
Two simplicial complexes

Let $S = K[x_1, \ldots, x_n]$. Denote by $sm(S)$ and $sm(I)$ the set of square-free monomials in $S$ and $I$ respectively. Denote by $sm(S)_d$ the set of square-free monomials of degree $d$ in $S$. 
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**A bijection between $sm(S)$ and $2^{[n]}$**

$$\sigma : x_{i_1}x_{i_2}\cdots x_{i_k} \mapsto \{i_1, i_2, \ldots, i_k\}.$$
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### Facet complex

$$\delta_{\mathcal{F}}(I) = \langle \sigma(G(I)) \rangle = \langle \{\sigma(g) \mid g \in G(I)\} \rangle.$$
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$$\delta_{\mathcal{F}}(I) = \langle \sigma(G(I)) \rangle = \langle \{\sigma(g) \mid g \in G(I)\} \rangle.$$  

**Stanley-Reisner complex**

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In other words, the Stanley-Reisner ideal of $\delta_{\mathcal{N}}(I)$ is $I$.  

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**Perfect sets and $f$-ideals**

**Outline**

**Introduction**

**Perfect sets and $f$-ideals of degree $d$**

$(n, 2)^{th}$ perfect number

**Structure of $V(n, 2)$**

**Further works**

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Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References

$f$-ideal

A square-free monomial ideal $I$ is called an $f$-ideal, if both
$\delta F(I)$ and $\delta N(I)$ have the same $f$-vector.

Background of $f$-ideal

Defined by G. Q. ABBASI, S. AHMAD, I. ANWAR and W. A. BAIG in [1];
The authors in [1] studied the properties of $f$-ideals of degree 2, and presented an interesting characterization of such ideals;
In [2], the authors generalized the characterization for $f$-ideals of degree $d$ ($d \geq 2$), though their main result seems to be a little bit inaccurate. See the following example.
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Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References

Background of $f$-ideal

Let $S = K[x_1, x_2, x_3, x_4, x_5]$, and let $I = \langle x_1 x_2 x_3, x_1 x_2 x_4, x_1 x_2 x_5, x_3 x_4 x_5, x_2 x_3 x_4 \rangle$. It is not hard to check that $I$ is an $f$-ideal. But the standard primary decomposition of $I$ is

$I = \langle x_2, x_5 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_1, x_4 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_3, x_4, x_5 \rangle$,

which shows that $I$ is not unmixed.
An example of $f$-ideal

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It is not hard to check that $I$ is an $f$-ideal. But the standard primary decomposition of $I$ is

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which shows that $I$ is not unmixed.
Four questions

Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References

Four questions

How to characterize $f$-ideals of degree $d$ directly?

How many $f$-ideals of degree $d$ are there in the polynomial ring $S = K[x_1, ..., x_n]$?

Is there any $f$-ideal which is not unmixed?

What can one say about $f$-ideals in general case?
Four questions

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- What can one say about $f$-ideals in general case?
Our work

Answer question (1);
Answer question (2) completely in the case $d = 2$;
Give a positive answer to question (3) in general case, and a negative answer in the case $d = 2$;
Give a preliminary answer to question (4).
Our work

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Perfect sets
and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets
and $f$-ideals
of degree $d$

$(n, 2)^{th}$
perfect
number

Structure of
$V(n, 2)$

Further
works

References

Some definitions

Upper generated and lower cover set

For a subset $A \subseteq \mathbb{S}(S)$, the upper generated set $\sqcup(A)$ of $A$ is defined by

\[ \sqcup(A) = \{ gx_i | g \in A, x_i \nmid g, 1 \leq i \leq n \} . \]

Dually, the lower cover set $\sqcap(A)$ of $A$ is defined by

\[ \sqcap(A) = \{ h | 1 \neq h, h = g/x_i \text{ for some } g \in A \text{ and some } x_i \text{ with } x_i \mid g \} . \]

Similarly, we define $\sqcup_2(A) = \sqcup(\sqcup(A))$, and $\sqcup_\infty(A) = \bigcup_{i=1}^{\infty} \sqcup_i(A)$, $\sqcap_\infty(A) = \bigcup_{i=1}^{\infty} \sqcap_i(A)$. 
Some definitions

Upper generated and lower cover set

For a subset $A \subseteq sm(S)$, the upper generated set $\sqcup(A)$ of $A$ is defined by

$$\sqcup(A) = \{gx_i \mid g \in A, x_i \nmid g, 1 \leq i \leq n\}.$$ 

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Similarly, we define $\sqcup^2(A) = \sqcup(\sqcup(A))$, and

$$\sqcup^\infty(A) = \bigcup_{i=1}^\infty \sqcup^i(A), \quad \sqcap^\infty(A) = \bigcup_{i=1}^\infty \sqcap^i(A).$$
Some definitions

Perfect set

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- upper perfect: If $\sqcup(A) = sm(S)_{d+1}$ holds;
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- upper perfect: If ⊔(A) = sm(S)_{d+1} holds;
- lower perfect: If ⊓(A) = sm(S)_{d-1} holds;
Some definitions

Perfect set

\( A \subseteq sm(S)_d \) is called

- upper perfect: If \( \uplus(A) = sm(S)_{d+1} \) holds;
- lower perfect: If \( \cap(A) = sm(S)_{d-1} \) holds;
- perfect: If \( A \) is both upper perfect and lower perfect.
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\[ A \subseteq sm(S)_d \] is called

- **upper perfect**: If \( \sqcup(A) = sm(S)_{d+1} \) holds;
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Some definitions

Homogeneous of degree $d$

A monomial ideal $I$ is called of degree $d$ (or alternatively, homogeneous of degree $d$), if all monomials in $G(I)$ have the same degree $d$. 
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A monomial ideal $I$ is called of degree $d$ (or alternatively, homogeneous of degree $d$), if all monomials in $G(I)$ have the same degree $d$.

If $I$ is an $f$-ideal of $S = K[x_1, \ldots, x_n]$, and homogeneous of degree $d$, we also call $I$ an $(n, d)^{th}$ $f$-ideal. Correspondingly, we can define an $(n, d)^{th}$ perfect sets.
Some definitions

Examples

Let $S = K[x_1, x_2, x_3, x_4]$. Consider the following three subsets of $sm(S)_2$: $A = \{x_1 x_2, x_1 x_3, x_1 x_4\}$, $B = \{x_1 x_2, x_1 x_3, x_2 x_3\}$, $C = \{x_1 x_2, x_3 x_4\}$. 

A is lower perfect but not upper perfect, since $x_2 x_3 x_4 \notin \bigcup A$; $B$ is upper perfect but not lower perfect, since $x_4 \notin \bigcap B$; $C$ is perfect.
Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References

Some definitions

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- $B$ is upper perfect but not lower perfect, since $x_4 \notin \sqcap(B)$;
- $C$ is perfect.
Main result of this part

Characterization of \((n, d)^{th}\) \(f\)-ideals

Let \(S = K[x_1, \ldots, x_n]\), and let \(I\) be a square-free monomial ideal of \(S\) of degree \(d\) with the minimal generating set \(G(I)\). Then \(I\) is an \(f\)-ideal if and only if the followings hold:
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- \(|G(I)| = \frac{1}{2} C_n^d;\)
Main result of this part

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- \(|G(I)| = \frac{1}{2}C_n^d\);
- \(G(I)\) is an \((n, d)^{th}\) perfect set.
Characterization of \((n, d)^{th}\) \(f\)-ideals

Let \(S = K[x_1, \ldots, x_n]\), and let \(I\) be a square-free monomial ideal of \(S\) of degree \(d\) with the minimal generating set \(G(I)\). Then \(I\) is an \(f\)-ideal if and only if the followings hold:

- \(|G(I)| = \frac{1}{2} C_{n}^{d};\)

- \(G(I)\) is an \((n, d)^{th}\) perfect set.
How to construct an \((n, d)^{th}\) \(f\)-ideal

1. Find an \((n, d)\)th perfect set \(A\), such that \(|A| \leq \frac{1}{2} C d^n\);
2. Choose \(D \subseteq \text{sm}(S) d \setminus A\) randomly, such that \(|D| = \frac{1}{2} C d^n - |A|\);
3. Let \(I\) be the ideal generated by \(A \cup D\);
   \(I\) is an \((n, d)^{th}\) \(f\)-ideal.
How to construct an \((n, d)^{th}\) \(f\)-ideal

- Find an \((n, d)^{th}\) perfect set \(A\), such that \(|A| \leq \frac{1}{2} C_n^d;\)
idea

How to construct an \((n, d)^{th}\) \(f\)-ideal

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How to construct an $$(n, d)^{th} f$$-ideal

- Find an $$(n, d)^{th}$$ perfect set $$A$$, such that \(|A| \leq \frac{1}{2} C_n^d$$;
- Choose $$D \subseteq sm(S)_d \setminus A$$ randomly, such that $$|D| = \frac{1}{2} C_n^d - |A|$$;
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1. Find an \((n, d)^{th}\) perfect set \(A\), such that \(|A| \leq \frac{1}{2} C_n^d\);

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Perfect sets and $f$-ideals

Jin Guo

Outline
Introduction
Perfect sets and $f$-ideals of degree $d$
$(n,2)^{th}$ perfect number
Structure of $V(n,2)$
Further works
References

How to construct an $(n, d)^{th}$ $f$-ideal

- Find an $(n, d)^{th}$ perfect set $A$, such that $|A| \leq \frac{1}{2} C_n^d$;

- Choose $D \subseteq sm(S)_d \setminus A$ randomly, such that $|D| = \frac{1}{2} C_n^d - |A|$;

- Let $I$ be the ideal generated by $A \cup D$;

- $I$ is an $(n, d)^{th}$ $f$-ideal.
How to find an \((n, d)^{th}\) perfect set

For a general \(d \geq 2\), it is not easy to find an \((n, d)^{th}\) perfect set, but it is not hard when \(d = 2\).
How to find an \((n, d)^{th}\) perfect set

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How to find an \((n, d)^{th}\) perfect set

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**How to find an \((n, 2)^{th}\) perfect set**

- Divide \([n]\) into two part \(B\) and \(C\) (Actually, \(C = \overline{B}\));
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For a general $d \geq 2$, it is not easy to find an $(n, d)^{th}$ perfect set, but it is not hard when $d = 2$.

**How to find an $(n, 2)^{th}$ perfect set**

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How to find an \((n, d)^{th}\) perfect set

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How to find an \((n, 2)^{th}\) perfect set

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(n, d)th perfect number

Definition

The least number among cardinalities of (n, d)th perfect sets, denoted by $N_{(n,d)}$. 
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Let $k$ be a positive integer, and let $n \geq 4$. Then the perfect number $N_{(n,2)}$ is given by the following rules:

- If $n = 2k$, then $N_{(n,2)} = C_k^2 + C_k^2 = k^2 - k$;
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The least number among cardinalities of (n, d)th perfect sets, denoted by N_{(n,d)}.

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Let k be a positive integer, and let n ≥ 4. Then the perfect number N_{(n,2)} is given by the following rules:

- If n = 2k, then N_{(n,2)} = \binom{2k}{k} + \binom{2k}{k} = k^2 - k;
- If n = 2k + 1, then N_{(n,2)} = \binom{2k}{k} + \binom{2k}{k+1} = k^2.
Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Definition

The least number among cardinalities of $(n, d)^{th}$ perfect sets, denoted by $N(n, d)$.

$(n, 2)^{th}$ perfect number

Let $k$ be a positive integer, and let $n \geq 4$. Then the perfect number $N(n, 2)$ is given by the following rules:

- If $n = 2k$, then $N(n, 2) = C_k^2 + C_k^2 = k^2 - k$;
- If $n = 2k + 1$, then $N(n, 2) = C_k^2 + C_{k+1}^2 = k^2$. 
Existence of \((n, 2)^{th}\) \(f\)-ideal

Existence

\[ V(n, 2) \neq \emptyset \text{ if and only if } 2 \mid C_n^2, \text{ i.e., if and only if } n = 4k \text{ or } n = 4k + 1 \text{ for some positive integer } k. \]
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Note that: When \(n = 4k\)

- \(N_{(n,2)} = 4k^2 - 2k\) and \(C_n^2/2 = 4k^2 - k;\)
Existence of \((n, 2)^{th}\) \textit{f}-ideal

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- \(N_{(n,2)} = 4k^2 - 2k\) and \(C_n^2/2 = 4k^2 - k\);

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### Existence

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Some notations

Two Part Complete Structure

For a subset $B$ of $\mathbb{N}$, denote $W_B = \{x_i x_j | i, j \in B \text{ or } i, j \in B\}$; Clearly $W_B = W_B$ holds, and $W_B$ is an $(n, 2)$th perfect set; A subset $A$ of $S(n, 2)$ is called satisfying Two Part Complete Structure, abbreviated as TPCS, if there exists a $B \subseteq \mathbb{N}$, such that $W_B \subseteq A$; If further $|B| = l$, then $A$ is called satisfying $l$th TPCS; An $f$-ideal $I$ is called of $l$ type, if $G(I)$ satisfies $l$th TPCS; Denote by $W_l$ the set of $f$-ideals of $l$ type in $S(n, 2)$.
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Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References

$\ell$ type

Question:

Let $S = K[x_1, x_2, x_3, x_4, x_5]$. Consider the ideal $I = \langle x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5, x_1 x_5 \rangle$. It is direct to check that $I$ is an $f$-ideal. But $I$ is not of $\ell$ type for any $\ell$.

Such kind of $f$-ideal of $K[x_1, x_2, x_3, x_4, x_5]$ is called $C_5$ ($5$-cycle).
Question:

Is there any $f$-ideal who is of no $l$ type?

Example

Let $S = K[x_1, x_2, x_3, x_4, x_5]$. Consider the ideal $I = \langle x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5, x_1 x_5 \rangle$. It is direct to check that $I$ is an $f$-ideal. But $I$ is not of $l$ type for any $l$. Such kind of $f$-ideal of $K[x_1, x_2, x_3, x_4, x_5]$ is called $C_5$ (5-cycle).
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- Such kind of $f$-ideal of $K[x_1, x_2, x_3, x_4, x_5]$ is called $C_5$ (5-cycle).
Another question:
Another question:
Is there any other $f$-ideal who is of no $l$ type?
Bijection $\tau$

$\tau$: From $2^{sm(S)_2}$ to the set of Graphs with $n$ vertices.
Bijection $\tau$

$\tau$: From $2^{sm(S)}_2$ to the set of Graphs with $n$ vertices.

- $A \subseteq sm(S)_2$, where $S = K[x_1, \ldots, x_n]$;
Bijection $\tau$

$\tau$: From $2^{sm(S)_2}$ to the set of Graphs with $n$ vertices.

- $A \subseteq sm(S)_2$, where $S = K[x_1, \ldots, x_n]$;
- $T = \tau(A)$ is a graph whose vertices are $v_1, \ldots, v_n$;
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- $v_iv_j \in E(T)$ holds if and only if $x_ix_j \in A$, where $E(T)$ is the edge set of $T$. 

Example: Let $S = K[x_1, x_2, x_3, x_4, x_5]$, and let $I = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1 \rangle$.

Note that $\tau(G(I)) = v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ is a 5-cycle.
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Note that $\tau(G(I)) = v_1 - v_2 - v_3 - v_4 - v_5 - v_1$ is a 5-cycle.
Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets

and $f$-ideals

degrees $d$

$(n,2)^{th}$ perfect
number

Structure of

$V(n,2)$

Further works

References

-----

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Perfect set

Let $A \subseteq \mathcal{S}(S)_2$. Then the followings hold:

- $A$ is upper perfect if and only if $\omega(\tau(A)) \leq 2$ holds, where $\tau(A)$ is the complement graph of $\tau(A)$.
- $A$ is lower perfect if and only if for each $i \in \{n\}$, $d(v_i) < n - 1$ holds in the graph $\tau(G(I))$.

If $I$ is an $(n, 2)^{th}$ f-ideal, then $I$ is of $l$ type for some $1 \leq l \leq \lfloor n/2 \rfloor$ if and only if $\tau(G(I))$ is a bipartite graph.
Perfect set

Let $A \subseteq sm(S)_2$. Then the followings hold:

- $A$ is upper perfect if and only if $\omega(\overline{\tau(A)}) \leq 2$ holds, where $\tau(A)$ is the complement graph of $\tau(A)$.
Perfect set

Let $A \subseteq sm(S)_2$. Then the followings hold:

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- $A$ is lower perfect if and only if for each $i \in [n]$, $d(v_i) < n - 1$ holds in the graph $\tau(A)$. 

Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals of degree $d$

$(n, 2)^{th}$ perfect number

Structure of $V(n, 2)$

Further works

References
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translation from combinatorics to graph theory
Perfect sets and $f$-ideals

Jin Guo

Outline

Introduction

Perfect sets and $f$-ideals

of degree $d$

$(n, 2)^{th}$

perfect number

Structure of $V(n, 2)$

Further works

References

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$l$ type

If $I$ is an $(n, 2)^{th}$ $f$-ideal, then $I$ is of $l$ type for some $1 \leq l \leq \lfloor n/2 \rfloor$ if and only if $\tau(G(I))$ is a bipartite graph.
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Four conditions

$I$ is an $(n, 2)^{th}$ $f$-ideal which is not of $l$ type for any $l$, if and only if $\tau(G(I))$ satisfies the following four conditions (abbreviated as FC in what follows):

1. For each $i \in [n]$, $d(v_i) < n - 1$ holds in $\tau(G(I))$.
2. $|E(\tau(G(I)))| = C_2^{2n^2}$.
3. $\omega(\tau(G(I))) = 2$.
4. $\tau(G(I))$ is not a bipartite graph.
Four conditions

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Main result

Structure of $V(n, 2)$

If $n \neq 5$, then $V(n, 2) = \bigcup_{l=1}^{\lfloor n/2 \rfloor} W_l$, which is a mutually disjoint union of the $W_l$'s.
Main result

Structure of $V(n, 2)$

If $n \neq 5$, then $V(n, 2) = \bigcup_{l=1}^{\lfloor n/2 \rfloor} W_l$, which is a mutually disjoint union of the $W_l$'s.

**Proof:** Note that $V(n, 2) = \bigcup_{l=1}^{\lfloor n/2 \rfloor} W_l$ holds true, if and only if each $f$-ideal is of $l$ type for some $l$; and the latter holds if and only if, there is no graph satisfying the FC. We will show that a graph will not satisfy condition (3) if it satisfies conditions (2) and (4), except for the case $n = 5$. 
Main result

Assume that $T$ is a graph satisfying conditions (2) and (4). Since $T$ is not a bipartite graph, there exists at least an odd cycle in $T$. Assume that $D$ is a minimal odd cycle of $T$, with $|V(D)| = 2i + 1$. Note that $\omega(T) = 2$, so $i \geq 2$. Denote by $|E(D)|$ the edge number of the subgraph induced on $D$, and denote by $|E(B, C)|$ the number of edges, each of which has end vertices in $B$ and $C$ respectively. It is clear that

$$|E(T)| = |E(D)| + |E(T \setminus D)| + |E(D, T \setminus D)|.$$ \hspace{1cm} \text{holds.}$$

Note that $|E(D)| = 2i + 1$ holds, since $D$ is a minimal cycle. Since there exists no triangles in $T$, it is not hard to see that

$$|E(D, T \setminus D)| \leq (n - 2i - 1)i$$ \hspace{1cm} \text{holds, since $D$ is an odd cycle. We will discuss $|E(T \setminus D)|$ in the following two subcases:}
Main result

If \( n = 2k \) for some positive \( k \), then \( |V(T \setminus D)| = 2k - 2i - 1 \) holds. It follows from Turan theorem that \( |E(T \setminus D)| \leq (k - i)(k - i - 1) \) hold, hence we get

\[
|E(T)| = |E(D)| + |E(T \setminus D)| + |E(D, T \setminus D)|
\leq (2i + 1) + (2k - 2i - 1)i + (k - i)(k - i - 1) = k^2 - k - i^2 + 2i + 1.
\]

Note that \( C^2_{2n}/2 = k^2 - k/2 \), thus

\[
C^2_{2n}/2 - |E(T)| \geq k/2 + i^2 - 2i - 1 = k/2 + (i - 1)^2 - 2
\]
holds. Since \( i \geq 2 \) and \( 2k > 2i + 1 \), \( C^2_{2n}/2 - |E(T)| > 0 \) holds. This shows that there is no graph satisfying FC when \( n = 2k \).
Main result

If \( n = 2k + 1 \), then \( |V(T \setminus D)| = 2k - 2i \) holds. Again by Turan theorem, \( |E(T \setminus D)| \leq (k - i)^2 \) holds, hence we have

\[
|E(T)| = |E(D)| + |E(T \setminus D)| + |E(D, T \setminus D)|
\]

\[
\leq (2i + 1) + (2k - 2i)i + (k - i)^2 = k^2 - i^2 + 2i + 1.
\]

Note that \( C_n^2/2 = k^2 + k/2 \), thus

\[
C_n^2/2 - |E(T)| \geq k/2 + i^2 - 2i - 1 = k/2 + (i - 1)^2 - 2
\]

holds true. Then we have \( C_n^2/2 - |E(T)| \geq 0 \), since \( i \geq 2 \) and \( k \geq i \) hold by assumption. Note further that the equality holds if and only if \( k = i = 2 \). Thus in this case, there is no graph satisfying FC except \( n = 5 \). This completes the proof.
Let $k$ be a positive integer. Then the following equalities hold true:

- $V(n, 2) = \bigcup_{0 \leq i \leq \sqrt{k}} W_{2k-i}$, if $n = 4k$;
Structure of $V(n, 2)$

Let $k$ be a positive integer. Then the following equalities hold true:

1. $V(n, 2) = \bigcup_{0 \leq i \leq \sqrt{k}} W_{2k-i}$, if $n = 4k$;
2. $V(n, 2) = \bigcup_{0 \leq i \leq \sqrt{1+4k-1}/2} W_{2k-i}$, if $n = 4k + 1 (k \neq 1)$;
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- $V(n, 2) = \emptyset$, if $n = 4k + 2$ or $n = 4k + 3$. 
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4. $V(n, 2) = \emptyset$, if $n = 4k + 2$ or $n = 4k + 3$. 


Cardinality of $V(n, 2)$

Let $k$ be a positive integer. Then the following equalities hold true:

- $|V(n, 2)| = \frac{1}{2} \binom{2k}{4k} \binom{k}{4k^2} + \sum_{1 \leq i \leq \sqrt{k}} \binom{2k-i}{4k} \binom{k-i^2}{4k^2-i^2}$, if $n = 4k$;
Cardinality of $V(n, 2)$

Let $k$ be a positive integer. Then the following equalities hold true:

- $|V(n, 2)| = \frac{1}{2} C_{4k}^{2k} C_{4k^2}^{k} + \sum_{1 \leq i \leq \sqrt{k}} C_{4k}^{2k-i} C_{4k^2-i^2}^{k-i^2}$, if $n = 4k$;

- $|V(n, 2)| = \sum_{0 \leq i \leq \frac{\sqrt{1+4k}-1}{2}} C_{4k+1}^{2k-i} C_{4k^2+2k-i-i^2}^{k-i-i^2}$, if $n = 4k + 1$ ($k \neq 1$).
Cardinality of $V(n, 2)$

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- $|V(n, 2)| = \frac{1}{2} \binom{2k}{4k} \binom{k}{4k^2} + \sum_{1 \leq i \leq \sqrt{k}} \binom{2k-i}{4k} \binom{k-i^2}{4k^2-i^2}$, if $n = 4k$;

- $|V(n, 2)| = \sum_{0 \leq i \leq \frac{\sqrt{1+4k-1}}{2}} \binom{2k-i}{4k+1} \binom{k-i^2}{4k^2+2k-i^2}$, if $n = 4k + 1 (k \neq 1)$;

- $|V(n, 2)| = 72$, if $n = 5$;
Let $k$ be a positive integer. Then the following equalities hold true:

- $|V(n, 2)| = \frac{1}{2} C_{4k}^{2k} C_{4k^2}^{k} + \sum_{1 \leq i \leq \sqrt{k}} C_{4k}^{2k-i} C_{4k^2-i^2}^{k-i^2}$, if $n = 4k$;

- $|V(n, 2)| = \sum_{0 \leq i \leq \sqrt{1+4k+1} - 1} C_{4k+1}^{2k-i} C_{4k^2+2k-i^2}^{k-i^2-i^2}$, if $n = 4k + 1 (k \neq 1)$;

- $|V(n, 2)| = 72$, if $n = 5$;

- $|V(n, 2)| = 0$, if $n = 4k + 2$ or $n = 4k + 3$. 

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Let $k$ be a positive integer. Then the following equalities hold true:

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- $|V(n, 2)| = \sum_{0 \leq i \leq \frac{\sqrt{1+4k} - 1}{2}} C_{4k+1}^{2k-i} C_{4k^2+2k-i-i^2}^k$, if $n = 4k + 1$ ($k \neq 1$);

- $|V(n, 2)| = 72$, if $n = 5$;

- $|V(n, 2)| = 0$, if $n = 4k + 2$ or $n = 4k + 3$. 
In general, an $f$-ideal may be not unmixed. But when $d = 2$, we have:
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**$f$-ideals of degree 2 is unmixed**

If $I$ is an $f$-ideal, then $I$ is unmixed.
Other results

In general, an $f$-ideal may be not unmixed. But when $d = 2$, we have:

**$f$-ideals of degree 2 is unmixed**

If $I$ is an $f$-ideal, then $I$ is unmixed.

Corresponding to $V(n, 2) \neq \emptyset$, we have:

**$V(n, d) \neq \emptyset$**

For any integer $d \geq 2$ and any integer $n \geq d + 2$ such that $2 \mid C^d_n$, $V(n, d) \neq \emptyset$. 
Other results

Denote $G(I) = \bigcup_{i=1}^{k} G_{d_i}$, in which $G_{d_i}$ consists of the generators of degree $d_i$. 
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### $f$-ideals in general case

Let $I$ be a square-free monomial ideal of $S = K[x_1, \ldots, x_n]$, with the minimal generating set $G(I) = \bigcup_{i=1}^{k} G_{d_i}$. Then $I$ is an $f$-ideal if and only if

$$|G_l| = \frac{1}{2} \left( C_n^l - |\bigcup_{d_i > l} \left( \bigcap_{d_i}^{d_i - l} (G_{d_i}) \right) | - |\bigcup_{d_i < l} \left( \bigcup_{l-d_i}^{l-d_i} (G_{d_i}) \right) | \right)$$

holds for each $l \in [n]$. 

Other results

unmixed $f$-ideals

$I$ is an $(n, d)^{th}$ unmixed $f$-ideal if and only if

$|G(I)| = C_{n}^{d}/2;$
Other results

**unmixed \( f \)-ideals**

\( I \) is an \((n, d)^{th}\) unmixed \( f \)-ideal if and only if
- \( |G(I)| = C_n^d / 2; \)
- \( G(I) \) is perfect;
unmixed $f$-ideals

$I$ is an $(n, d)^{th}$ unmixed $f$-ideal if and only if
- $|G(I)| = C^d_n/2$;
- $G(I)$ is perfect;
- $\langle \sigma(u) \mid u \in sm(S)_d \setminus G(I) \rangle$ is a $d$-flag complex.
Other results

unmixed $f$-ideals

$I$ is an $(n, d)^{th}$ unmixed $f$-ideal if and only if

- $|G(I)| = C^d_n/2$;
- $G(I)$ is perfect;
- $sm(S)_d \setminus G(I)$ is lower perfect.
Other results

unmixed \( f \)-ideals

\( I \) is an \((n, d)^{th}\) unmixed \( f \)-ideal if and only if

1. \( |G(I)| = \frac{C_n^d}{2} \);
2. \( G(I) \) is perfect;
3. \( sm(S)_d \setminus G(I) \) is lower perfect.

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Further works

Questions

- How to calculate the perfect number $N_{(n,d)}$?
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- What about nonhomogeneous $f$-ideal?
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- What about the structure of $V(n,d)$?
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References


Thank you!!!

Author: Guo Jin
Address: Department of Mathematics
Shanghai Jiaotong University
Shanghai, 200240, China
Email: guojinecho@163.com