On a class of squarefree monomial ideals of linear type

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Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \ldots, x_n]$ a polynomial ring of $n$ variables. A monomial $x^a := x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} \in S$ is squarefree if each $a_i \in \{0, 1\}$. Its degree is $\deg(x^a) = a_1 + \cdots + a_n$. An ideal $I$ of $S$ is squarefree if it can be (minimally) generated by a (finite and unique) set of squarefree monomials. A squarefree monomial ideal of degree 2 (i.e., a quadratic monomial ideal) is a squarefree monomial ideal whose minimal monomial generators are all of degree 2.
Two ways to connect squarefree monomial ideals to combinatorial objects

1. $I$ is the Stanley-Reisner ideal of some simplicial complex.
2. $I$ is the facet ideal of another simplicial complex. Equivalently, $I$ is the (hyper)edge ideal of some clutter.

**Definition**

Let $V$ be a finite set. A clutter $C$ with vertex set $V(C) = V$ consists of a set $E(C)$ of subsets of $V$, called the edges of $C$, with the property that no edge contains another. Clutters are special hypergraphs.

- Squarefree ideals of degree 2 $\Leftrightarrow$ (finite simple) graphs.
- Squarefree ideals of higher degree $\Leftrightarrow$ clutters of higher dimension.
Examples

Example (1)

\[ \langle x_1 x_2, x_2 x_5, x_3 x_5, x_1 x_3, x_1 x_4 \rangle \subset \mathbb{K}[x_1, \ldots, x_5]. \]

Example (2)

\[ \langle x_1 x_2 x_5 x_6, x_2 x_3 x_7 x_8, x_3 x_4 x_9 x_{10}, x_1 x_4 x_{11} x_{12}, x_3 x_8 x_9 \rangle \subset \mathbb{K}[x_1, \ldots, x_{12}]. \]
Interplay between combinatorics and commutative algebra

Commutative algebra $\Rightarrow$ combinatorics

E.g., Richard Stanley’s proof of the **Upper Bound Conjecture** for simplicial spheres by means of the theory of Cohen-Macaulay rings.

Combinatorics $\Rightarrow$ commutative algebra

E.g., if $G$ is a graph and each of its connected components has at most one odd cycle (i.e., each component either has no cycle, or has no even cycle), then its edge ideal $I(G)$ is of **linear type**.
Let $S$ be a Noetherian ring and $I$ an $S$-ideal. The **Rees algebra of $I$** is the subring of the ring of polynomials $S[t]$

$$\mathcal{R}(I) := S[lt] = \bigoplus_{i \geq 0} I^i t^i.$$ 

Analogously, one has $\text{Sym}(I)$, the **symmetric algebra of $I$** which is obtained from the tensor algebra of $I$ by imposing the commutative law.

There is a canonical surjection $\Phi: \text{Sym}(I) \twoheadrightarrow \mathcal{R}(I)$. When the canonical map $\Phi$ is an isomorphism, $I$ is called an ideal of **linear type**.
The symmetric algebra $\text{Sym}(I)$ is equipped with an $S$-Module homomorphism $\pi: I \rightarrow \text{Sym}(I)$ which solves the following universal problem. For a commutative $S$-algebra $B$ and any $S$-module homomorphism $\varphi: I \rightarrow B$, there exists a unique $S$-algebra homomorphism $\Phi: \text{Sym}(I) \rightarrow B$ such that the diagram

$$
\begin{array}{ccc}
I & \xrightarrow{\varphi} & B \\
\downarrow{\pi} & & \uparrow{\Phi} \\
\text{Sym}(I) & & 
\end{array}
$$

is commutative.
Suppose $I = \langle f_1, \ldots, f_s \rangle$ and consider the $S$-linear presentation
\[\psi : S[T] := S[T_1, \ldots, T_s] \to S[lt]\]
defined by setting $\psi(T_i) = f_it$. Since this map is homogeneous, the kernel $J = \bigoplus_{i \geq 1} J_i$ is a graded ideal; it will be called the \textbf{defining ideal} of $\mathcal{R}(I)$ (with respect to this presentation). Since the linear part $J_1$ generates the defining ideal of $\text{Sym}(R)$, $I$ is of linear type if and only if $J = \langle J_1 \rangle$.

The maximal degree in $T$ of the minimal generators of the defining ideal $J$ is called the \textbf{relation type} of $I$. 
Example of defining ideals

Example (3)

Let $S = \mathbb{K}[x_1, \ldots, x_7]$ and $I$ be the ideal of $S$ generated by $f_1 = x_1x_2x_3$, $f_2 = x_2x_4x_5$, $f_3 = x_5x_6x_7$ and $f_4 = x_3x_6x_7$. Then the defining ideal is minimally generated by $x_3T_3 - x_5T_4$, $x_6x_7T_1 - x_1x_2T_4$, $x_6x_7T_2 - x_2x_4T_3$, $x_4x_5T_1 - x_1x_3T_2$ and $x_4T_1T_3 - x_1T_2T_4$.

Check for $x_4T_1T_3 - x_1T_2T_4$:

$$x_4T_1T_3 \mapsto x_4(x_1x_2x_3t)(x_5x_6x_7t),$$
$$x_1T_2T_4 \mapsto x_1(x_2x_4x_5t)(x_3x_6x_7t).$$

This minimal generator of the defining ideal is of degree 2 in $T$. Thus the ideal $I$ is not of linear type. Indeed, its relation type is 2.
The defining ideal of squarefree monomial ideals are always binomial, i.e., are generated by binomials.

**Theorem (Taylor)**

Suppose $I$ is minimally generated by monomials $f_1, \ldots, f_s$. Let $\mathcal{I}_k$ be the set of non-decreasing sequence of integers in $\{1, 2, \ldots, s\}$ of length $k$. If $\alpha = (i_1, i_2, \ldots, i_k) \in \mathcal{I}_k$, set $f_{\alpha} = f_{i_1} \cdots f_{i_k}$ and $T_{\alpha} = T_{i_1} \cdots T_{i_k}$. For every $\alpha, \beta \in \mathcal{I}_k$, set

$$T_{\alpha, \beta} = \frac{f_{\beta}}{\gcd(f_{\alpha}, f_{\beta})} T_{\alpha} - \frac{f_{\alpha}}{\gcd(f_{\alpha}, f_{\beta})} T_{\beta}.$$ 

Then the defining ideal $J$ is generated by these $T_{\alpha, \beta}$’s with $\alpha, \beta \in \mathcal{I}_k$ and $k \geq 1$. 
How to compute?

- **Q**: How to compute the defining ideal? **A**: Gröbner basis theory.
- **Q**: How to check the minimality? **A**: Gröbner basis theory.

**Websites:**
- Macaulay2 → http://www.math.uiuc.edu/Macaulay2/
- Singular → http://www.singular.uni-kl.de/
- CoCoA System → http://cocoa.dima.unige.it/

**Example (2, continued)**

\[
\langle x_1x_2x_5x_6, x_2x_3x_7x_8, x_3x_4x_9x_{10}, x_1x_4x_{11}x_{12}, x_3x_8x_9 \rangle \subset K[x_1, \ldots, x_{12}]
\]

is of linear type.
Macaulay 2 codes for Example 2

```
[10:31:27] [2013SJTU]$ M2
Macaulay2, version 1.6
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : S=QQ[x_1..x_12]
o1 = S
o1 : PolynomialRing

i2 : I = monomialIdeal(x_1*x_2*x_5*x_6,x_2*x_3*x_7*x_8,x_3*x_4*x_9*x_10,
x_1*x_4*x_11*x_12,x_3*x_8*x_9)
o2 = monomialIdeal (x x x , x x x x , x x x x , x x x , x x x x , x x x )
    1 2 5 6   2 3 7 8   3 8 9   3 4 9 10   1 4 11 12
o2 : MonomialIdeal of S

i3 : isLinearType ideal I
o3 = true
```
Singular codes for Example 3

> LIB "reesclos.lib";
> ring S=0,(x(1..7)),dp;
> ideal I=x(1)*x(2)*x(3),x(2)*x(4)*x(5), x(5)*x(6)*x(7),
    x(3)*x(6)*x(7);
> list L=ReesAlgebra(I);
> def Rees=L[1];
> setring Rees;
> Rees;
// characteristic : 0
// number of vars : 11
// block 1 : ordering dp
//    : names x(1) x(2) x(3) x(4) x(5) x(6) x(7)
//        U(1) U(2) U(3) U(4)
// block 2 : ordering C
> ker;
ker[1]=x(3)*U(3)-x(5)*U(4)
ker[2]=x(4)*U(1)*U(3)-x(1)*U(2)*U(4)
ker[3]=x(6)*x(7)*U(2)-x(2)*x(4)*U(3)
ker[4]=x(6)*x(7)*U(1)-x(1)*x(2)*U(4)
ker[5]=x(4)*x(5)*U(1)-x(1)*x(3)*U(2)
> ideal NewVars=U(1),U(2),U(3),U(4);
> ideal LI=reduce(ker,std(NewVars^2));
> LI;
LI[1]=x(3)*U(3)-x(5)*U(4)
LI[2]=0
LI[3]=x(6)*x(7)*U(2)-x(2)*x(4)*U(3)
LI[4]=x(6)*x(7)*U(1)-x(1)*x(2)*U(4)
LI[5]=x(4)*x(5)*U(1)-x(1)*x(3)*U(2)
> reduce(ker, std(LI));
_[1]=0
_[2]=x(4)*U(1)*U(3)-x(1)*U(2)*U(4)
_[3]=0
_[4]=0
_[5]=0
Theorem (Villarreal)

Let $\mathcal{G}$ be a connected graph and $I = I(\mathcal{G})$ its edge ideal. Then $I$ is an ideal of linear type if and only if $\mathcal{G}$ is a tree or $\mathcal{G}$ has a unique cycle of odd length. This result is independent of the characteristic of the base field $K$.

Example (4)

The Stanley-Reisner ring of the real projective plane is Cohen-Macaulay if and only if the characteristic of the base field is not 2.
**Definition (Generator graph)**

Let $I$ be a squarefree monomial ideal whose minimal monomial generating set is $\{ f_1, \ldots, f_s \}$. Let $G$ be a graph whose vertices $v_i$ corresponds to $f_i$ respectively and two vertices $v_i$ and $v_j$ are adjacent if and only if the two monomials $f_i$ and $f_j$ have a non-trivial GCD. This graph $G$ is called the **generator graph** of $I$.

**Theorem (Fouli and Lin)**

*When $I$ is a squarefree monomial ideal and the generator graph of $I$ is the graph of a disjoint union of trees and graphs with a unique odd cycle, then $I$ is an ideal of linear type.*
Let $I$ be a monomial ideal in $S = \mathbb{K}[x_1, \ldots, x_n]$. Let $x_{n+1}$ be a new variable with $S' = \mathbb{K}[x_1, \ldots, x_n, x_{n+1}]$. Then $I$ is a squarefree monomial ideal if and only if $I' = I \cdot x_{n+1}$ is so. And $I$ is of linear type if and only if $I'$ is so. Indeed, $I'$ and $I$ will have essentially identical defining ideals. However, the generator graph of $I'$ is a complete graph.
Leaves and quasi-forests

Definition
Let \( \Delta \) be a clutter. The edge \( F \) of \( \Delta \) is a leaf of \( \Delta \) if there exists an edge \( G \) such that \( (H \cap F) \subseteq (G \cap F) \) for all edges \( H \in \Delta \). The edge \( G \) is called a branch or joint of \( F \).

Definition
A clutter \( \Delta \) is called a quasi-forest if there exists a total order of the edges \( \{ F_1, \ldots, F_m \} \) such that \( F_i \) is a leaf of the sub-clutter \( \langle F_1, \ldots, F_i \rangle \) for all \( i = 1, \ldots, m \). This order is called a leaf order of the quasi-forest. A connected quasi-forest is called a quasi-tree.
Forests

Definition

A (simplicial) forest is a clutter $\Delta$ which enjoys the property that for every subset $\{F_{i_1}, \ldots, F_{i_q}\}$ of $\mathcal{F}(\Delta)$ the sub-clutter $\langle F_{i_1}, \ldots, F_{i_q} \rangle$ of $\Delta$ has a leaf. A tree is a forest which is connected.

Facts

1. Edge ideals of forests are always of linear type.
2. Edge ideals of quasi-forests are not necessarily of linear type.
Example

This is a quasi-tree, but not a tree. Its edge ideal

\[ \langle x_1x_2x_3x_4, x_1x_4x_5, x_1x_2x_8, x_2x_3x_7, x_3x_4x_6 \rangle \]

is not of linear type.
An edge $F$ of the clutter $\Delta$ is called a **good leaf** if this $F$ is a leaf of each sub-clutter $\Gamma$ of $\Delta$ to which $F$ belongs. An order $F_1, \ldots, F_s$ of the edges is called a **good leaf order** if $F_i$ is a good leaf of $\langle F_1, \ldots, F_i \rangle$ for each $i = 1, \ldots, s$.

It is known that a clutter is a forest if and only if it has a good leaf order.
Theorem (Shen)

Suppose $\Delta$ is a clutter which is obtained from the clutter $\Delta'$ by adding a good leaf. If the edge ideal of $\Delta'$ is of linear type, then the edge ideal of $\Delta$ also shares this property.

Tools: Gröbner basis. This result reproves the fact that the edge ideals of forests are of linear type.

Question: Is the converse true?

Suppose $\Delta$ is a clutter which is obtained from the clutter $\Delta'$ by adding a good leaf. If the edge ideal of $\Delta$ is of linear type, does the edge ideal of $\Delta'$ also share this property?
Theorem (Conca and De Negri)

If $I$ is a monomial ideal which is generated by an $M$-sequence, then $I$ is of linear type.

Facts

The monomial ideal $I$ is generated by an $M$-sequence if and only if this $I$ is of forest type (in the sense of Soleyman Jahan and Zheng). In particular, when $I$ is squarefree, $I$ is generated by an $M$-sequence if and only if it is the edge ideal of a simplicial forest, and this $M$-sequence corresponds to the good leaf order of the forest.
Simplicial cycles

**Definition**

A clutter $\Delta$ is called a **simplicial cycle** or simply a **cycle** if $\Delta$ has no leaf but every nonempty proper sub-clutter of $\Delta$ has a leaf.

This definition is more restrictive than the classic definition of (hyper)cycles of hypergraphs due to Berge.

**Fact**

If $\Delta$ is a simplicial cycle. Then

1. either the generator graph of the edge ideal of $\Delta$ is a cycle, or,
2. $\Delta$ is a cone over such a structure.
**Definition**

Let $\mathcal{V}$ be the class of clutters minimal with respect to the following properties:

- Disjoint simplicial cycles of odd lengths are in $\mathcal{V}$, with simplexes being considered as simplicial cycles of length 1.
- $\mathcal{V}$ is closed under the operation of attaching good leaves.

We shall call $\mathcal{V}$ the *Villarreal class*. When a clutter $\Delta$ is in $\mathcal{V}$, we say $\Delta$ and its edge ideal $I(\Delta)$ are *of Villarreal type*.

**Theorem (Shen)**

*Squarefree monomial ideals of Villarreal type are of linear type (but not vice versa).*
$\Delta$ is not a simplicial cycle and has no leaves. It is not of Villarreal type, but its edge ideal is of linear type. On the other hand, the ideal of $\tilde{\Delta}$ is not of linear type.
This $\Delta'$ is a simplicial cycle of length 4. Both $\Delta$ and $\tilde{\Delta}$ are obtained from $\Delta$ by attaching new edges. The $\tilde{G}$ in $\tilde{\Delta}$ introduces a new vertex while the $\Gamma$ in $\Delta$ does not. The $\Gamma$ is a patch attached to $\Delta'$ connecting the adjacent edges $F_2$ and $F_3$.

**Theorem (Shen)**

Suppose $\Delta'$ is a simplicial cycle of even length and $\Delta$ is obtained from $\Delta'$ by attaching a patch. Then the edge ideal of $\Delta$ is of linear type.


Further reading

Thank you!

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