Pure-cycle Hurwitz factorizations and multi-noded rooted trees

by Rosena Ruoxia Du

East China Normal University

Combinatorics Seminar, SJTU
August 29, 2013

This is joint work with Fu Liu.
PART I:

Definitions and Backgrounds
**Hurwitz’s problem**

**Definition 1.** Given integers \( d \) and \( r \), and \( r \) partitions \( \lambda^1, \ldots, \lambda^r \vdash d \), a **Hurwitz factorization** of type \((d, r, (\lambda^1, \ldots, \lambda^r))\) is an \( r \)-tuple \((\sigma_1, \ldots, \sigma_r)\) satisfying the following conditions:

(i) \( \sigma_i \in S_d \) has cycle type (or is in the conjugacy class) \( \lambda^i \), for every \( i \);

(ii) \( \sigma_1 \cdots \sigma_r = 1 \);

(iii) \( \sigma_1, \ldots, \sigma_r \) generate a transitive subgroup of \( S_d \).
Definition 1. Given integers $d$ and $r$, and $r$ partitions $\lambda^1, \ldots, \lambda^r \vdash d$, a Hurwitz factorization of type $((d, r, (\lambda^1, \ldots, \lambda^r)))$ is an $r$-tuple $(\sigma_1, \ldots, \sigma_r)$ satisfying the following conditions:

(i) $\sigma_i \in \mathfrak{S}_d$ has cycle type (or is in the conjugacy class) $\lambda^i$, for every $i$;

(ii) $\sigma_1 \cdots \sigma_r = 1$;

(iii) $\sigma_1, \ldots, \sigma_r$ generate a transitive subgroup of $\mathfrak{S}_d$.

Definition 2. The Hurwitz number $h(d, r, (\lambda^1, \ldots, \lambda^r))$ is the number of Hurwitz factorizations of type $((d, r, (\lambda^1, \ldots, \lambda^r)))$ divided by $d!$. 

**Hurwitz’s problem**
**Hurwitz’s problem**

**Definition 1.** Given integers $d$ and $r$, and $r$ partitions $\lambda^1, \ldots, \lambda^r \vdash d$, a *Hurwitz factorization* of type $(d, r, (\lambda^1, \ldots, \lambda^r))$ is an $r$-tuple $(\sigma_1, \ldots, \sigma_r)$ satisfying the following conditions:

(i) $\sigma_i \in \mathfrak{S}_d$ has cycle type (or is in the conjugacy class) $\lambda^i$, for every $i$;

(ii) $\sigma_1 \cdots \sigma_r = 1$;

(iii) $\sigma_1, \ldots, \sigma_r$ generate a transitive subgroup of $\mathfrak{S}_d$.

**Definition 2.** The *Hurwitz number* $h(d, r, (\lambda^1, \ldots, \lambda^r))$ is the number of Hurwitz factorizations of type $(d, r, (\lambda^1, \ldots, \lambda^r))$ divided by $d!$.

**Question:** What is the Hurwitz number $h(d, r, (\lambda^1, \ldots, \lambda^r))$?

This question originally arises from geometry: Hurwitz number counts the number of degree-$d$ covers of the projective line with $r$ branch points where the monodromy over the $i$th branch point has cycle type $\lambda^i$.
The pure-cycle case

A number of people (Hurwitz, Goulden, Jackson, Vakil ...) have studied Hurwitz numbers. However, they often restricted their attention to the case where all but one or two $\sigma_i$'s are transpositions.
The pure-cycle case

A number of people (Hurwitz, Goulden, Jackson, Vakil ...) have studied Hurwitz numbers. However, they often restricted their attention to the case where all but one or two $\sigma_i$'s are transpositions.

We consider instead the pure-cycle case. This means each $\lambda^i$ has the form $(e_i, 1, \ldots, 1)$, for some $e_i \geq 2$, or equivalently, each $\sigma_i$ is an $e_i$ cycle. In this case, we use the notation $h(d, r, (e_1, \ldots, e_r))$ for the Hurwitz number.

We also focus on the genus-0 case, which simply means that

$$2d - 2 = \sum_{i=1}^{r} (e_i - 1).$$
The pure-cycle case

A number of people (Hurwitz, Goulden, Jackson, Vakil ...) have studied Hurwitz numbers. However, they often restricted their attention to the case where all but one or two $\sigma_i$'s are transpositions.

We consider instead the pure-cycle case. This means each $\lambda^i$ has the form $(e_i, 1, \ldots, 1)$, for some $e_i \geq 2$, or equivalently, each $\sigma_i$ is an $e_i$ cycle. In this case, we use the notation $h(d, r, (e_1, \ldots, e_r))$ for the Hurwitz number.

We also focus on the genus-0 case, which simply means that

$$2d - 2 = \sum_{i=1}^{r} (e_i - 1).$$

**Example 3.** Let $d = 5$, $r = 4$, $(e_1, e_2, e_3, e_4) = (2, 2, 3, 5)$. One can check that

$$((2 \ 3), (4 \ 5), (1 \ 3 \ 5), (5 \ 4 \ 3 \ 2 \ 1))$$

is a genus-0 pure-cycle Hurwitz factorization.

(Genus-0: $2d - 2 = 8 = \sum_{i=1}^{4} (e_i - 1) = 1 + 1 + 2 + 4.$)
Previous results on the pure-cycle case

**Lemma 4** (Liu-Osserman). *In the genus-0 pure-cycle case, when* $r = 3$,

\[ h(d, 3, (e_1, e_2, e_3)) = 1. \]
Previous results on the pure-cycle case

**Lemma 4** (Liu-Osserman). *In the genus-0 pure-cycle case, when \( r = 3 \),

\[
h(d, 3, (e_1, e_2, e_3)) = 1.
\]

**Theorem 5** (Liu-Osserman). *In the genus-0 pure-cycle case, when \( r = 4 \),

\[
h(d, 4, (e_1, e_2, e_3, e_4)) = \min\{e_i(d + 1 - e_i)\}
\]
We study a special case of genus-0 pure-cycle Hurwitz factorizations: when one of the $e_i$ is $d$. W.L.O.G, we assume $e_r = d$.

Then the “genus-0” condition becomes:

$$2d - 2 = \sum_{i=1}^{r} (e_i - 1) \quad \Rightarrow \quad \sum_{i=1}^{r-1} (e_i - 1) = d - 1.$$ 

Since $\sigma_r$ is a $d$-cycle, $\langle \sigma_1, \ldots, \sigma_r \rangle$ is automatically transitive in $\mathfrak{S}_d$.

Moreover,

$$\sigma_1 \ldots \sigma_r = 1 \quad \iff \quad \sigma_1 \ldots \sigma_{r-1} = \sigma_r^{-1}.$$
Factorizations of a $d$-cycle
Definition 6. Assume $d, r \geq 1, e_1, \ldots, e_{r-1} \geq 2$ are integers satisfying $\sum_{i=1}^{r-1} (e_i - 1) = d - 1$. Fix a $d$-cycle $\tau \in S_d$. We say $(\sigma_1, \ldots, \sigma_{r-1})$ is a factorization of $\tau$ of type $(e_1, \ldots, e_{r-1})$ if the followings are satisfied:

i. For each $i$, $\sigma_i$ is an $e_i$-cycle in $S_d$.

ii. $\sigma_1 \cdots \sigma_{r-1} = \tau$. 
Factorizations of a $d$-cycle

**Definition 6.** Assume $d, r \geq 1, e_1, \ldots, e_{r-1} \geq 2$ are integers satisfying $\sum_{i=1}^{r-1} (e_i - 1) = d - 1$. Fix a $d$-cycle $\tau \in S_d$. We say $(\sigma_1, \ldots, \sigma_{r-1})$ is a factorization of $\tau$ of type $(e_1, \ldots, e_{r-1})$ if the followings are satisfied:

i. For each $i$, $\sigma_i$ is an $e_i$-cycle in $S_d$.

ii. $\sigma_1 \cdots \sigma_{r-1} = \tau$.

**Example 7.** Let $d = 5$, $r = 4$, $(e_1, e_2, e_3) = (2, 2, 3)$, $\tau = (1 2 3 4 5)$, $\sigma_1 = (2 3)$, $\sigma_2 = (4 5)$, $\sigma_3 = (1 3 5)$. It is easy to check that $(\sigma_1, \sigma_2, \sigma_3)$ is a factorization of $\tau$ of type $(2, 2, 3)$:

$$(2 3)(4 5)(1 3 5) = (1 2 3 4 5).$$
Factorizations of a $d$-cycle

**Definition 6.** Assume $d, r \geq 1, e_1, \ldots, e_{r-1} \geq 2$ are integers satisfying $\sum_{i=1}^{r-1}(e_i - 1) = d - 1$. Fix a $d$-cycle $\tau \in S_d$. We say $(\sigma_1, \ldots, \sigma_{r-1})$ is a factorization of $\tau$ of type $(e_1, \ldots, e_{r-1})$ if the followings are satisfied:

i. For each $i$, $\sigma_i$ is an $e_i$-cycle in $S_d$.

ii. $\sigma_1 \cdots \sigma_{r-1} = \tau$.

**Example 7.** Let $d = 5, r = 4, (e_1, e_2, e_3) = (2, 2, 3), \tau = (1\ 2\ 3\ 4\ 5), \sigma_1 = (2\ 3), \sigma_2 = (4\ 5), \sigma_3 = (1\ 3\ 5)$. It is easy to check that $(\sigma_1, \sigma_2, \sigma_3)$ is a factorization of $\tau$ of type $(2, 2, 3)$:

$$(2\ 3)(4\ 5)(1\ 3\ 5) = (1\ 2\ 3\ 4\ 5).$$

**Question 8.** Given a $d$-cycle $\tau$ and integers $e_1, \ldots, e_{r-1} \geq 2$, how many factorizations are there of $\tau$ of type $(e_1, \ldots, e_{r-1})$?
**Main Result**

**Theorem 9.** Suppose \( \sum_{i=1}^{r-1} (e_i - 1) = d - 1 \). Then the number of factorizations of a \( d \)-cycle of type \((e_1, \ldots, e_{r-1})\) is

\[
\text{fac}(d, r; e_1, \ldots, e_{r-1}) = d^{r-2}.
\]
Main Result

**Theorem 9.** Suppose $\sum_{i=1}^{r-1} (e_i - 1) = d - 1$. Then the number of factorizations of a \(d\)-cycle of type \((e_1, \ldots, e_{r-1})\) is

$$\text{fac}(d, r; e_1, \ldots, e_{r-1}) = d^{r-2}.$$ 

**Example 10.** There are \(3 = 3^1\) factorizations of \((1 \ 2 \ 3)\) of type \((2, 2)\):

\[(12)(23) \quad (23)(13) \quad (13)(12)\]
Main Result

Theorem 9. Suppose \( \sum_{i=1}^{r-1} (e_i - 1) = d - 1 \). Then the number of factorizations of a \( d \)-cycle of type \( (e_1, \ldots, e_{r-1}) \) is

\[
fac(d, r; e_1, \ldots, e_{r-1}) = d^{r-2}.
\]

Example 10. There are \( 3 = 3^1 \) factorizations of \( (1 2 3) \) of type \( (2, 2) \):

\[
(12)(23) \quad (23)(13) \quad (13)(12)
\]

Example 11. There are \( 25 = 5^2 \) factorizations of \( (1 2 3 4 5) \) of type \( (2, 2, 3) \):

\[
(12)(23)(345) \quad (23)(34)(451) \quad (34)(45)(512) \quad (45)(51)(123) \quad (51)(12)(234) \\
(23)(13)(345) \quad (34)(24)(451) \quad (45)(35)(512) \quad (51)(41)(123) \quad (12)(52)(234) \\
(13)(12)(345) \quad (24)(23)(451) \quad (35)(34)(512) \quad (41)(45)(123) \quad (52)(51)(234) \\
(12)(34)(245) \quad (23)(45)(351) \quad (34)(51)(412) \quad (45)(12)(523) \quad (51)(23)(134) \\
(34)(12)(245) \quad (45)(23)(351) \quad (51)(34)(412) \quad (12)(45)(523) \quad (23)(51)(134)
\]
When \( e_1 = \cdots = e_{r-1} = 2 \), from \( \sum_{i=1}^{r-1} (e_i - 1) = d - 1 \) we have \( d = r \). Then Theorem 9 gives the following well-known result:

**Corollary 12.** The number of factorizations of a \( d \)-cycle into \( d - 1 \) transpositions is \( d^{d-2} \).
When $e_1 = \cdots = e_{r-1} = 2$, from $\sum_{i=1}^{r-1} (e_i - 1) = d - 1$ we have $d = r$. Then Theorem 9 gives the following well-known result:

**Corollary 12.** The number of factorizations of a $d$-cycle into $d - 1$ transpositions is $d^{d-2}$.

Note that $d^{d-2}$ is also the number of trees on $d$ vertices. Different bijective proofs of this result were given by Dénes (1959), Moszkowski (1989), Goulden-Pepper (1993) and Goulden-Yong (2002).
When \( e_1 = \cdots = e_{r-1} = 2 \), from \( \sum_{i=1}^{r-1} (e_i - 1) = d - 1 \) we have \( d = r \). Then Theorem 9 gives the following well-known result:

**Corollary 12.** The number of factorizations of a \( d \)-cycle into \( d - 1 \) transpositions is \( d^{d-2} \).

Note that \( d^{d-2} \) is also the number of trees on \( d \) vertices. Different bijective proofs of this result were given by Dénes (1959), Moszkowski (1989), Goulden-Pepper (1993) and Goulden-Yong (2002).

**Main Idea to prove Theorem 9:** Construct a class of combinatorial objects that are counted by \( d^{r-2} \), and then find a bijection between factorizations and them.
PART II:

Multi-noded Rooted Trees
Definition of Multi-noded Rooted Trees

Definition 13. Suppose $f_0, f_1, \ldots, f_n$ are positive integers and $S = \{s_1, \ldots, s_n\}$. We say $G$ is a \textit{multi-noded rooted tree} on $S \cup \{0\}$ \textit{of vertex data} $(f_0, f_1, \ldots, f_n)$ if we have the followings:

(i) The vertex set of $G$ is $S \cup \{0\}$.

(ii) For each vertex $s_i$, it includes $f_i$ ordered nodes (by convention, $s_0 := 0$).

(iii) Considering only vertices and edges, $G$ is a rooted tree with root 0, but in addition each edge is connected to a particular node of the parent vertex.

We denote by $\mathcal{MR}_S(f_0, f_1, \ldots, f_n)$ the set of multi-noded rooted trees.

Example 14. A multi-noded rooted tree of vertex data $(1, 1, 2, 1, 2, 2, 3, 3, 1, 4)$:

![Multi-noded Rooted Tree Diagram]
Theorem 15. \(|\mathcal{MR}_S(f_0, f_1, \ldots, f_n)| = f_0 (\sum_{i=0}^{n} f_i)^{n-1} \).
Counting Multi-noded Rooted Trees

Theorem 15. \(|\mathcal{MR}_S(f_0, f_1, \ldots, f_n)| = f_0 \left(\sum_{i=0}^{n} f_i\right)^{n-1}\).

Corollary 16. Suppose \(\sum_{j=1}^{r-1} (e_j - 1) = d - 1\). Then

\[|\mathcal{MR}_S(1, e_1 - 1, \ldots, e_{r-1} - 1)| = d^{r-2}.\]
Theorem 15. $|\mathcal{MR}_S(f_0, f_1, \ldots, f_n)| = f_0 \left( \sum_{i=0}^{n} f_i \right)^{n-1}$. 

Corollary 16. Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. Then 

$$|\mathcal{MR}_S(1, e_1 - 1, \ldots, e_{r-1} - 1)| = d^{r-2}.$$
Theorem 15. $|\mathcal{MR}_S(f_0, f_1, \ldots, f_n)| = f_0 \left(\sum_{i=0}^{n} f_i\right)^{n-1}$.

Corollary 16. Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. Then

$|\mathcal{MR}_S(1, e_1 - 1, \ldots, e_{r-1} - 1)| = d^{r-2}$. 
Theorem 15. \[ |\mathcal{MR}_S(f_0, f_1, \ldots, f_n)| = f_0 \left( \sum_{i=0}^{n} f_i \right)^{n-1}. \]

Corollary 16. Suppose \( \sum_{j=1}^{r-1} (e_j - 1) = d - 1 \). Then
\[ |\mathcal{MR}_S(1, e_1 - 1, \ldots, e_{r-1} - 1)| = d^{r-2}. \]
Counting Multi-noded Rooted Trees

**Theorem 15.** \(|MR_S(f_0, f_1, \ldots, f_n)| = f_0 (\sum_{i=0}^{n} f_i)^{n-1}.

**Corollary 16.** Suppose \(\sum_{j=1}^{r-1} (e_j - 1) = d - 1\). Then
\[
|MR_S(1, e_1 - 1, \ldots, e_{r-1} - 1)| = d^{r-2}.
\]
Theorem 15. $|MR_S(f_0, f_1, \ldots, f_n)| = f_0 (\sum_{i=0}^{n} f_i)^{n-1}$.

Corollary 16. Suppose $\sum_{j=1}^{r-1} (e_j - 1) = d - 1$. Then

$$|MR_S(1, e_1 - 1, \ldots, e_{r-1} - 1)| = d^{r-2}.$$
Counting Multi-noded Rooted Trees

Theorem 15. \(|\mathcal{MR}_S(f_0, f_1, \ldots, f_n)| = f_0 \left( \sum_{i=0}^{n} f_i \right)^{n-1} .

Corollary 16. Suppose \(\sum_{j=1}^{r-1} (e_j - 1) = d - 1 .\) Then

\(|\mathcal{MR}_S(1, e_1 - 1, \ldots, e_{r-1} - 1)| = d^{r-2} .

\[
\begin{pmatrix}
  s_3 & s_9 & s_2 & s_9 & s_3 & 0 & s_9 & s_5 & 0 \\
  1 & 3 & 2 & 1 & 1 & 1 & 3 & 1 & 1
\end{pmatrix}.
\]
PART III:

Bijection between Factorizations and Multi-noded Rooted Trees
Factorization Graphs
A factorization of $\tau = (1 \ 2 \ 3 \ \cdots \ 20)$ of type $(2, 3, 2, 3, 4, 4, 2, 5)$:

$$(10 \ 11)(14 \ 15 \ 19)(1 \ 19)(3 \ 4 \ 5)(1 \ 2 \ 13)(15 \ 16 \ 17 \ 18)(7 \ 8 \ 9 \ 11)(19 \ 20)(2 \ 5 \ 6 \ 11 \ 12)$$
A factorization of $\tau = (1\ 2 \cdots \ 20)$ of type $(2, 3, 2, 3, 4, 4, 2, 5)$:

$$(10\ 11)(14\ 15\ 19)(1\ 19)(3\ 4\ 5)(1\ 2\ 13)(15\ 16\ 17\ 18)(7\ 8\ 9\ 11)(19\ 20)(2\ 5\ 6\ 11\ 12)$$

The factorization graph associated to this factorization is:
A factorization of $\tau = (1 2 \cdots 20)$ of type $(2, 3, 2, 3, 4, 4, 2, 5)$:


The factorization graph associated to this factorization is:

**Facts:**
1. $G$ is a bipartite graph on $S \cup [d]$.
2. Any vertex $s_i$ has degree $e_i$. 

---

**Factorization Graphs**
Characterization of factorization graphs

Proposition 17. Suppose $\sum_{j=1}^{r-1}(e_j - 1) = d - 1$, and $G$ is a bipartite graph on $S \cup [d]$ such that vertex $s_i$ has degree $e_i$.

Then $G$ is a factorization graph associated to a factorization of $\tau$ of type $(e_1, \ldots, e_{r-1})$ if and only if $G$ satisfies the following conditions:

i. $G$ is a tree.

ii. For each $[d]$-vertex $\nu$ of $G$, suppose $\{s_{j_1} < s_{j_2} < \cdots < s_{j_t}\}$ are the vertices adjacent to $\nu$ in $G$. We get $t$ subtrees after removing $\nu$ and all its incident edges. Then

(a) The $[d]$-vertices of the $t$ subtrees partition $[d] \setminus \{\nu\}$ into contiguous pieces.

(b) If we order the pieces in counterclockwise order on $\tau$ starting from $\nu$, then the $m$-th piece is exactly the subtree that contains vertex $s_{j_m}$ for any $1 \leq m \leq t$.‌
A factorization of $\tau = (1 \, 2 \, \cdots \, 20)$:

A factorization of $\tau = (1 \ 2 \ \cdots \ 20)$:

$$(10 \ 11)(14 \ 15 \ 19)(1 \ 19)(3 \ 4 \ 5)(1 \ 2 \ 13)(15 \ 16 \ 17 \ 18)(7 \ 8 \ 9 \ 11)(19 \ 20)(2 \ 5 \ 6 \ 11 \ 12)$$

The factorization graph associated to a factorization of type $(2, 3, 2, 3, 3, 4, 4, 2, 5)$
Factorization Graphs to Labeled Multi-noded Rooted Trees

A factorization of \( \tau = (1 \ 2 \ \cdots \ 20) \):

\[
(10\ 11)(14\ 15\ 19)(1\ 19)(3\ 4\ 5)(1\ 2\ 13)(15\ 16\ 17\ 18)(7\ 8\ 9\ 11)(19\ 20)(2\ 5\ 6\ 11\ 12)
\]

The factorization graph associated to a factorization of type \((2, 3, 2, 3, 3, 4, 4, 2, 5)\)

A labelled multi-noded rooted tree of vertex data \((1, 1, 2, 1, 2, 2, 3, 3, 1, 4)\)
Remark

Goulden and Jackson considered more general factorizations of a $d$-cycle, where they allow $\sigma_i$ to be any cycle type, that is, $\sigma_i$ does not have to be a cycle. They gave a formula for the factorization number in this situation.
Remark

Goulden and Jackson considered more general factorizations of a $d$-cycle, where they allow $\sigma_i$ to be any cycle type, that is, $\sigma_i$ does not have to be a cycle. They gave a formula for the factorization number in this situation.

But their proof involves calculation of generating functions.
Remark

Goulden and Jackson considered more general factorizations of a $d$-cycle, where they allow $\sigma_i$ to be any cycle type, that is, $\sigma_i$ does not have to be a cycle. They gave a formula for the factorization number in this situation.

But their proof involves calculation of generating functions.

An equivalent *symmetrized* version of Theorem 9 was proved by Springer and Irving separately: e.g., when $(e_1, e_2, e_3) = (2, 2, 3)$, we only allow factorizations where the first and second cycles have length 2 and the third cycle has length 3. They included all factorizations with one 3-cycle and two 2-cycles.
Goulden and Jackson considered more general factorizations of a \( d \)-cycle, where they allow \( \sigma_i \) to be any cycle type, that is, \( \sigma_i \) does not have to be a cycle. They gave a formula for the factorization number in this situation.

But their proof involves calculation of generating functions.

An equivalent *symmetrized* version of Theorem 9 was proved by Springer and Irving separately: e.g., when \((e_1, e_2, e_3) = (2, 2, 3)\), we only allow factorizations where the first and second cycles have length 2 and the third cycle has length 3. They included all factorizations with one 3-cycle and two 2-cycles.

We believe that our proof is the first “de-symmetrized” direct bijective proof.