Hamiltnocity of prisms over graphs

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What is a prism over a graph?
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**Definition**

A graph $G$ is d-polyhedral if it is the 1-skeleton of a d-dimensional polytope.
Polyhedral graphs

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2. No characterization of $k$-polyhedral graphs for $k \geq 4$ is known
3. The complete graph $K_n$ is $k$-polyhedral for all $k \geq 4$.
4. The prisms over $k$-polyhedral graphs are $(k+1)$-polyhedral.
A k-polyhedral graph is k-connected.
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Every graph is an induced subgraph of k-polyhedral graphs for \( k \geq 4 \).
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Tutte’s theorem:
Polyhedral graphs

1. A k-polyhedral graph is k-connected.

2. Every graph is an induced subgraph of k-polyhedral graphs for $k \geq 4$.

3. Tutte’s theorem:

4. A 4-connected planar graph is Hamiltonian.
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Every graph is an induced subgraph of $k$-polyhedral graphs for $k \geq 4$.

Tutte’s theorem:

A 4-connected planar graph is Hamiltonian.

Note that a planar graph of order $n$ has at most $3n - 6$ edges, a sparse graph, so it should be surprising that it is Hamiltonian.
Prisms over cubic graphs

It turned out that planarity was not a factor: the prism over cubic, 3-connected graphs are Hamiltonian (Paulraja (1993), R.Cada, T. Kaiser, M.R. & Z. Ryjacek (2001))
It turned out that planarity was not a factor: the prism over cubic, 3-connected graphs are Hamiltonian (Paulraja (1993), R.Cada, T. Kaiser, M.R. & Z. Ryjacek (2001))

The main tool was the even-cactus:
The proof consists of two steps:

1. Every 2-connected subcubic graph has a spanning cactus.
2. Every 3-connected cubic graph has a spanning 2-connected bipartite subcubic graph.
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1. Every 2-connected subcubic graph has a spanning cactus.

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### Observation

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1. Cubic graphs with a factorization such that every two factors form a hamiltonian cycle.
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4. The duals of 4-dimensional stacked polytopes.
5. Petersen’s graph.
The dual of cyclic polytopes

The dual of a $d$-polytope $P$ is a $d$-polytope $P^*$ in which the facets of $P$ correspond to vertices of $P^*$ such that two vertices of $P^*$ are connected by an edge if and only if the two corresponding facets have a $d - 2$ dimensional face in common.
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The cyclic polytope in $R^4$ is obtained by taking the convex hull of $n$ points on the moment curve $\{(1, t, t^2, t^3)\}$. The graph of its dual can be described combinatorially using Gale’s evenness condition:
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$$V(G) = \{[i, j, k, m] \mid 1 \leq i < j < k < m \leq n\}$$

such that any integers $a, b \not\in \{i, j, k, m\}$ are separated by an even number of integers from $\{i, j, k, m\}$.

$$E(G) = \{(A, B) \mid |A \cap B| = 3\}.$$
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$$E(G) = \{(A, B) \mid |A \cap B| = 3\}.$$

It is not difficult to check that this graph is 4-regular and 4-connected.
We need to find two even cacti that share the even cycles, disjoint green edges that include all edges.
Petersen’s Hamiltonian decomposition

The prism over Petersen’s graph

Start with a $C_8$
Petersen's Hamiltonian decomposition

The first prism over Petersen’s graph

The first Cactus and the Hamiltonian cycle generated by it.
The complementary Cactus and the Hamiltonian cycle

1 – 5 – 4 – 10 – 6 – 7 — 8 – 9            3 – 2 – 1
                                                    |            |    
               9 – 2 – 1 – 6 – 10 – 4 – 5 – 8 – 7 – 3
What makes (made) a problem famous?
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Hamiltonicity of prisms over graphs
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Are prisms over 3-connected cubic graphs Hamilton decomposable?

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As a first step, it is listed among the 100 problems in Adrian Bondy’s new book.
What do we know

1. True for bipartite planar, cubic 3-connected graphs.

2. Not true for 2-connected cubic graphs, even planar.

3. Our strategy is to tackle this question "piece by piece".

4. In the next slides we shall explore some tools and examples of Hamilton decomposable families of prisms over cubic graphs.
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A Hamilton cycle over $K_4$.  

[Diagram of a Hamilton cycle over $K_4$.]
Hamilton Decompositions samples, the basics.

Hamiltonian decomposition of the prism over $K_4$.

The generalized Cacti associated with each cycle.
Can the given Hamiltonian cycle be the “blue-yellow” cycle for the Hamilton decomposition of the prism?

Can a Hamiltonian cycle in a cubic graph “help” us find a Hamiltonian decomposition in its prism?
Can the given Hamiltonian cycle be the “blue-yellow” cycle for the Hamilton decomposition of the prism?

Hamiltonian decomposition of prisms over Hamiltonian cubic graphs.

We wish to incorporate the green edges to get the Hamiltonian decomposition.
Can the given Hamiltonian cycle be the “blue-yellow” cycle for the Hamilton decomposition of the prism?
Prisms \((C_n \times K_2)\) are Hamiltonian, their prisms are Hamilton decomposable, but the Hamilton cycle can not be used as the “blue-yellow” cycle to decompose them.
Hamiltonian cubic graphs

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Question

1. Given a cycle \(C_{2n}\) and \(n\) diagonals. Is it possible to determine in polynomial time whether the diagonals can be split into two sets such that each set together with the Hamiltonian cycle will produce Hamiltonian cycles in the prism?
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**Question**

2. **Given a planar, Hamiltonian cubic graph and the hamiltonian cycle. Can the cycle be used as the “blue-yellow” cycle for the decomposition?**
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We proved that the prisms over halin graphs are Hamiltonian.
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Here we’ll be concerned with cubic Halin graphs, i.e. a binary tree plus a cycle through its leaves. For example, Petersen’s graph is such a graph.
Generalized Halin Graphs

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Conjecture

The prisms over cubic Halin graphs are Hamilton decomposable.
Halin representation of Petersen’s graph.
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One Hamiltonian cycle in the prism.

Halin representation of Petersen’s graph. 
Second Hamiltonian cycle in the prism
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All 3 connected cubic graphs can be generated from $K_4$ by $H$ or $A$ operations.
H-free cubic graphs

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All 3 connected cubic graphs can be generated from $K_4$ by $H$ or $A$ operations.

**Conjecture**

*The prism of all 3-connected cubic graphs generated from $K_4$ by $A$ operations are Hamilton decomposable.*
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Definition (slightly less Hamiltonian)

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In a talk in the conference, Mark Ellingham gave a survey of k-walks in graphs.

Jackson and Wormald pointed out in 1990 the following sharp “hierarchy:”

1-walk (Hamilton) \(\Rightarrow\) 2-tree (traceable) \(\Rightarrow\) 2-walk \(\Rightarrow\) 3-tree \(\Rightarrow\) ...
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Do prisms fit in this hierarchy?

We noted: $2$-tree $\subset$ Hamiltonian prism $\subset$ $2$-walk and all inclusions are sharp.

Example

1. In 1967 D. Barnette proved that all $3$-polytopes have a spanning $3$-tree.
2. In 1994 Z. Gao and B. Richter improved it by proving that $3$-polytopes have a $2$-walk.
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Conjecture
The prisms over 3-polytopes are Hamiltonian
Opportunities

The prisms provide us with many opportunities to revisit Hamiltonian problems, results and even resuscitate “dead” conjectures.

1. Is the mid-level graph Hamiltonian? (open)
2. We proved that the prism over the mid-level graph is Hamiltonian. (2005)
3. 4-connected 4-regular graphs are Hamiltonian? (Nash Williams)
4. NO! (Meredith's construction)
5. Are the prisms over 4-connected 4-regular graphs Hamiltonian? I conjecture YES!
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Open problems

The prisms over graphs open the door for many Hamilton cycles related problems. The number of related results is growing. We highlighted three open problems:

1. Are the prisms over 3-connected, cubic graph Hamilton decomposable?
2. Are the prisms over 3-connected planar graphs hamiltonian?
3. Are the prisms over 4-connected, 4-regular graphs Hamiltonian?
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Dense graphs

There is a very large number of papers devoted to Hamilton cycles in “dense graphs:” namely graphs with $cn^2$ edges. These problems usually start with Dirac’s or Ore’s theorem:

Theorem (Dirac’s)

If $\delta(G) \geq \frac{n}{2}$, $G$ a graph of order $n$ then $G$ is Hamiltonian.

Theorem (Ore’s)

If for any two vertices $u$, $v$ of a graph $G$ of order $n$, not connected by an edge, $\deg_G(u) + \deg_G(v) \geq n$ then $G$ is Hamiltonian.
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A sample of problems

One type of problem is whether by adding a “few” edges (usually increasing the degree requirements) we can get some more specific Hamiltonian cycles.

For instance, we can ask when for a given fixed number of vertices $v_{i_1}, v_{i_2}, \ldots v_{i_k}$ can we have a Hamiltonian cycle in $G$ in which these vertices appear in this order in the cycle?
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For instance, we can ask when for a given fixed number of vertices \( v_{i_1}, v_{i_2}, \ldots v_{i_k} \) can we have a Hamiltonian cycle in \( G \) in which these vertices appear in this order in the cycle? When can we specify a path of length \( k \) in \( G \) and find a Hamiltonian cycle that contains this path?
Problems on Hamiltonian cycles in prisms

In the prism paradigm we can ask the same questions for sparse graphs. For instance:

1. Is it true that for any four vertices $u_1, u_2, u_3, u_4$ of the cubic, 3-connected graph $G$, one can find a Hamiltonian cycle in the prism over $G$ in which these vertices appear in this order? How far can we extend this? ($5, 6, \ldots$ some fraction of $n$)
Problems on Hamiltonian cycles in prisms

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1. Is it true that for any four vertices $u_1, u_2, u_3, u_4$ of the cubic, 3-connected graph $G$, one can find a Hamiltonian cycle in the prism over $G$ in which these vertices appear in this order? How far can we extend this? (5, 6, \ldots \text{ some fraction of } n)

2. Same for the generalized Halin graphs.
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3. (Enomoto) Is it true that for every pair of vertices $u, v$ of $G$, a 3-connected cubic graph, one can find a Hamiltonian cycle in the prism over $G$ in which $u$ and $v$ appear at distance $n$ from each other?
**A Sample of recent results**

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$CL\frac{4n}{3} - \frac{4}{3}$ means repeatedly adding an edge between two vertices not connected by an edge if the sum of their degrees is $\geq \frac{4n}{3} - \frac{4}{3}$ (D. Král and L. Stacho, 2004)
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4. **Kneser Graphs** The prism over $K(4k + 1, 2k)$ is hamiltonian. (L. R. Bueno, P. Horàk, 2011)
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Thank you.