The Terwilliger algebra of the Johnson scheme $J(N,D)$ from the viewpoint of group representations

Ying-Ying Tan (谭莹莹)
Joint work with Xiaoye Liang, Tasturo Ito

Anhui University

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1 Introduction

2 Group representation theory

3 Terwilliger theory about \((P & Q)\) polynomial scheme
Irreducible character of $G$ on $X_i$

- $\Omega$: set, $|\Omega| = N$.
- $X_i = \binom{\Omega}{i} = \{x \subset \Omega \mid |x| = i\}$, $X := X_D$.
- $G := \text{Sym}(\Omega)$: the symmetric group on $\Omega$.
- $\pi_i$: the permutation character of $G$ acting on $X_i$.

**Fact**

Let $\chi_0 = 1_G$ be identity character, $\chi_i = \pi_i - \pi_{i-1}$ ($1 \leq i \leq \frac{N}{2}$). Then

1. $\chi_0, \chi_1, \cdots, \chi_{\frac{N}{2}}$ are distinct irreducible characters of $G$.
2. $\pi_i = \chi_0 + \chi_1 + \cdots + \chi_i$ ($0 \leq i \leq \frac{N}{2}$).
Let $V = \mathbb{C}^X \simeq \bigoplus_{x \in X} \mathbb{C}x$ be the standard module.

$\blacktriangleright$ $G$ acts on $X := X_D = (\Omega_D)$,

the permutation character $\pi := \pi_D = \chi_0 + \chi_1 + \cdots + \chi_D$.

$\blacktriangleright$ $V$ is the permutation $G$-module affording $\pi$,

$V = V_0 \oplus V_1 \oplus \cdots \oplus V_D$, where $V_i$ is the irreducible $G$-module

affording $\chi_i$. 
Centralizer algebra of permutation representations

The centralizer algebra of $G$

$$\text{Hom}_G(V, V) = \{f \in \text{End}(V) | \ f(av) = af(v) \ \forall v \in V, \forall a \in G\}.$$  

**Fact**

Let $\mathcal{A} := \text{Hom}_G(V, V) \subset \text{End}(V) \cong \text{Mat}_X(\mathbb{C})$,

- $\mathcal{A} = \langle A_0, A_1, \cdots, A_D \rangle$, where $\forall x, y \in X$,

$$\begin{cases} 
1 & \text{if } |x \cap y| = D - i, \\
0 & \text{otherwise}.
\end{cases}$$

- $\mathcal{A} = \langle E_0, E_1, \cdots, E_D \rangle$, where

$$E_i : V = \bigoplus_{x \in X} \mathbb{C}x \longrightarrow V_i \text{ projection}$$
Orbits of group action

$G$ acts on $X = \binom{D}{0}$ transitively.

- $G$ acts on $X \times X$,

$G$-orbits on $X \times X$: $R_0, R_1, \ldots, R_D$, where

$$R_i = \{(x, y) | |x \cap y| = D - i, \forall x, y \in X\},$$

which correspond to $A_0, A_1, \ldots, A_D$.

Let $H := G_{x_0}$ be the stabilizer of $x_0$ in $G$, $x_0$ is a fixed base point.

- $H$ acts on $X$,

$H$-orbits on $X$: $R_0(x_0), R_1(x_0), \ldots, R_D(x_0)$,

where $R_i(x_0) = \{x \in X | |x \cap x_0| = D - i\}$,

$$V = V_0^* \oplus V_1^* \oplus \ldots \oplus V_D^*, \text{ where } V_i^* = \bigoplus_{x \in R_i(x_0)} \mathbb{C} x.$$
Johnson Scheme

**Definition**

Let \( \Omega \) be a set of cardinality \( N \). Let

\[
X = \binom{\Omega}{D} = \{ x \subset \Omega \mid |x| = D \},
\]

\[R_i \ni (x, y) \iff |x \cap y| = D - i.\]

Then \( \mathcal{X} = (X, \{ R_i \}_{0 \leq i \leq D}) \) is an association scheme which is called Johnson scheme \( J(N, D) \).
Three algebras

- \( \mathfrak{A} := \text{Hom}_G(V,V) \) the centralizer algebra of \( G \)
  
  \[ = \langle A_0, A_1, \cdots, A_D \rangle \]  
  Bose-Mesner algebra of \( J(N,D) \)

- \( \mathfrak{A}^* = \langle E_0^*, E_1^*, \cdots, E_D^* \rangle \) Dual Bose-Mesner algebra of \( J(N,D) \)
  
  where \( E_i^* : V \rightarrow V_i^* \) projection.

- \( T := \langle \mathfrak{A}, \mathfrak{A}^* \rangle \) Terwilliger algebra of \( J(N,D) \)
  
  \[ = \langle E_0, E_1, \cdots, E_D, E_0^*, E_1^*, \cdots, E_D^* \rangle \subset \text{End}(V). \]

- \( S := \text{Hom}_H(V,V) \) the centralizer algebra of \( H = G_{x_0} \).
Question

- $\mathfrak{A}$, $S$ and $T$ are semi-simple algebras.
- $\mathfrak{A} \subset T \subseteq S$.

Question:

How large is the gap between $S$ and $T$ for $J(N,D)$?
Our result

Main Theorem

For Johnson scheme $J(N,D)(2D \leq N)$,

- if $N \neq 2D$, then $T = S$;
- if $N = 2D$, then $T \subset S$. 
Notations

- $\Omega$: set, $|\Omega| = N$, $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, $|\Omega_1| = D$, $|\Omega_2| = N - D$, ($2D \leq N$).

- $G = \text{Sym}(\Omega)$, $X = \binom{\Omega}{D}$, $X \ni x_0 = \Omega_1$ base point.

- $H = G_{x_0} \simeq \text{Sym}(\Omega_1) \times \text{Sym}(\Omega_2)$.

- $\pi_i^{(1)}$: the permutation character of $\text{Sym}(\Omega_1)$ on $\binom{\Omega_1}{D-i} \simeq \binom{\Omega_1}{i}$, $0 \leq i \leq \frac{D}{2}$.

- $\pi_i^{(2)}$: the permutation character of $\text{Sym}(\Omega_2)$ on $\binom{\Omega_2}{i}$, $0 \leq i \leq \frac{N-D}{2}$. 
Irreducible character of $H \cong \text{Sym}(\Omega_1) \times \text{Sym}(\Omega_2)$

$\pi_i^{(1)} = \chi_0^{(1)} + \chi_1^{(1)} + \cdots + \chi_i^{(1)}$

where $\chi_\alpha^{(1)}$ is an irreducible character of $\text{Sym}(\Omega_1)$ on $^{\Omega_1}_i$,

$0 \leq \alpha \leq \min\{i,D-i\}$.

$\pi_i^{(2)} = \chi_0^{(2)} + \chi_1^{(2)} + \cdots + \chi_i^{(2)}$

where $\chi_\beta^{(2)}$ is an irreducible character of $\text{Sym}(\Omega_2)$ on $^{\Omega_2}_i$,

$0 \leq \beta \leq \min\{i,N-D-i\}$.

$\chi_\alpha^{(1)} \chi_\beta^{(2)}$ is an irreducible character of $H \cong \text{Sym}(\Omega_1) \times \text{Sym}(\Omega_2)$

$0 \leq \alpha \leq \frac{D}{2}, \quad 0 \leq \beta \leq \min\{D,\frac{N-D}{2}\}$. 
Character of $H$ acting on $V_i$

Let $V = \mathbb{C}^X \simeq \bigoplus_{x \in X} \mathbb{C}x$ be the permutation $G$-module,

- $V = V_0 \oplus V_1 \oplus \cdots \oplus V_D$, where $V_i$ is the irreducible $G$-module affording $\chi_i$.

**Theorem**

Let $\chi_i|_H$ be the character of $H$ acting on $V_i$. For $0 \leq i \leq D$,

$$\chi_i|_H = \sum_{(\alpha, \beta) \in \Phi_i} \chi_{(1)}^{\alpha} \chi_{(2)}^{\beta},$$

where

$$\Phi_i = \{(\alpha, \beta) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | 0 \leq \alpha + \beta \leq i, i - N + D \leq \alpha - \beta \leq D - i\}.$$
Multiplicity of $\chi^{(1)}_\alpha \chi^{(2)}_\beta$ in $V_i$ — $Q$-structure of $J(N,D)$

Corollary

Let $\mu = \alpha + \beta$, $\mu + d = \min\{D,D - \alpha + \beta,N - D + \alpha - \beta\}$,

$$
(\chi_i, \chi^{(1)}_\alpha \chi^{(2)}_\beta)_H = \begin{cases} 
1 & \text{if } \mu \leq i \leq \mu + d \\
0 & \text{otherwise.}
\end{cases}
$$
Permutation character of $H$ acting on $V_i^*$

Let $V = \mathbb{C}^X \cong \bigoplus_{x \in X} \mathbb{C}x$ be the permutation $H$-module,

- $V = V_0^* \oplus V_1^* \oplus \ldots \oplus V_D^*$, where $V_i^* = \sum_{x \in R_i(x_0)} \mathbb{C}x$,

$H$-orbits $R_i(x_0) \cong (\Omega_{D-i}) \times (\Omega_i)$, $x_0$ is the base point.

**Lemma**

Let $\pi_{D-i}^{(1)}$ be the permutation character of $\text{Sym}(\Omega_1)$ on $(\Omega_{D-i})$ and $\pi_i^{(2)}$ the permutation character of $\text{Sym}(\Omega_2)$ on $(\Omega_i)$, then

(1) $\pi_{D-i}^{(1)} \pi_i^{(2)}$ is the permutation character of $H$ on $R_i(x_0)$,

(2) $\pi_{D-i}^{(1)} \pi_i^{(2)} = \sum_{0 \leq \alpha \leq \min\{i, D-i\}} \sum_{0 \leq \beta \leq \min\{i, N-D-i\}} \chi_{\alpha}^{(1)} \chi_{\beta}^{(2)}$. 
Multiplicity of $\chi^{(1)}_{\alpha} \chi^{(2)}_{\beta}$ in $V^*_i$ — $P$-structure of $J(N,D)$

Corollary

Let $\nu = \max\{\alpha, \beta\}$, $\nu + d = \min\{D - \alpha, N - D - \beta\}$,

$$(\pi_{D-i}^{(1)} \pi_i^{(2)}, \chi^{(1)}_{\alpha} \chi^{(2)}_{\beta}) = \begin{cases} 1 & \text{if } \nu \leq i \leq \nu + d \\ 0 & \text{otherwise.} \end{cases}$$
Homogeneous component decomposition for $H$

Let $\Lambda = \{(\alpha, \beta) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | 0 \leq \alpha \leq \frac{D}{2}, \quad 0 \leq \beta \leq \min\{D, \frac{N-D}{2}\}, 0 \leq \alpha + \beta \leq D\}$.

Homogeneous component decomposition for $H$

$$V = \bigoplus_{x \in X} Cx = \bigoplus_{(\alpha, \beta) \in \Lambda} V(\alpha, \beta),$$

where $V(\alpha, \beta)$ is the sum of irreducible $H$-modules affording the irreducible character $\chi_\alpha^{(1)} \chi_\beta^{(2)}$. 
Homogeneous component decomposition for $S$-module

**General fact**

Let $R = \mathbb{C}[H]$ be the group algebra of $H$ and $S = \text{Hom}_H(V, V)$.

- $R$ and $S$ are semi-simple.
- $V = \bigoplus_{(\alpha, \beta) \in \Lambda} V(\alpha, \beta)$ is the homogeneous component decomposition for $S$ as well.

Since $V$ is a faithful $S$-module,

every irreducible $S$-module appears in $V$.

<table>
<thead>
<tr>
<th>Isomorphism classes of irreducible $S$-modules</th>
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<tbody>
<tr>
<td>$\upharpoonright 1:1$</td>
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<tr>
<td>$(\alpha, \beta) \in \Lambda$</td>
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1st key theorem

Let $T$ be the Terwilliger algebra of $J(N,D)$ and $S = \text{Hom}_H(V,V)$. If $V \supset W$ is irreducible as an $S$-module, then $W$ is irreducible as a $T$-module.

Proof: by Kantor’s lemma.

(Inclusion mapping is of full rank.)

Next, we want to know

$$W \cong W' \text{ as irreducible } S\text{-modules}$$

\[\Downarrow ?\]

$$W \cong W' \text{ as irreducible } T\text{-modules}$$
Mapping in 2nd key theorem

$\Delta$

Let $\Delta$ be the set of ordered triples $(\nu, \mu, d)$ of non-negative integers $\nu, \mu, d$ that satisfy

$$0 \leq \frac{D-d}{2} \leq \nu \leq \mu \leq D - d \leq D,$$

$$d \in \{D - 2\nu, \min(D - \mu, N - D - 2\nu)\}.$$  

$\Lambda$

Let $\Lambda$ be the set of pairs $(\alpha, \beta)$ of non-negative integers $\alpha, \beta$ such that

$$0 \leq \alpha \leq \frac{D}{2},$$

$$0 \leq \beta \leq \min(D, \frac{N - D}{2}),$$

$$0 \leq \alpha + \beta \leq D.$$
Define a mapping \( \varphi : \Lambda \rightarrow \Delta, (\alpha, \beta) \mapsto (\nu, \mu, d) \) by

\[
\begin{align*}
\nu &= \text{max}(\alpha, \beta), \\
\mu &= \alpha + \beta, \\
d &= \begin{cases} 
D - 2\alpha & \text{if } \beta \leq \alpha, \\
D - \alpha - \beta & \text{if } 0 \leq \beta - \alpha \leq N - 2D, \\
N - D - 2\beta & \text{if } N - 2D \leq \beta - \alpha.
\end{cases}
\end{align*}
\]

**2nd key theorem** [X. Liang, Y. Tan and T. Ito, 2017]

1. The mapping \( \varphi : \Lambda \rightarrow \Delta \) is a bijection if \( N \neq 2D \).
2. Let \( \Lambda_0 \) be the subset of \( \Lambda \) consisting of \( (\alpha, \beta) \in \Lambda \) that satisfy \( \beta \leq \alpha \). Then the mapping \( \varphi|_{\Lambda_0} : \Lambda_0 \rightarrow \Delta \) is a bijection if \( N = 2D \).
Isomorphism classes of irreducible $S$-module

In case of $J(N,D)$,

$$\{\text{Isomorphism classes of irreducible } S\text{-modules}\}$$

$\uparrow$ General fact

$$\{\text{Isomorphism classes of irreducible } H\text{-modules}\}$$

$\downarrow$ $V = \bigoplus_{(\alpha,\beta) \in \Lambda} V(\alpha,\beta)$

$$\{(\alpha,\beta) \in \Lambda\}$$

$\uparrow$ Mapping in 2nd key theorem for $N \neq 2D$

$$\{(\nu,\mu,d) \in \Delta\}$$
Terwilliger theory about \((P & Q)\) polynomial scheme

**Theorem [Terwilliger, 1993]**

In case of Johnson scheme \(J(N,D)\), let \(W, W'\) denote irreducible \(T\)-module, then the following are equivalent.

1. \(W \simeq W'\) as \(T\)-module.
2. \(W, W'\) have the same \((\nu, \mu, d) \in \Delta\), where \(\nu\) is the dual-endpoint, \(\mu\) is the endpoint, \(d\) is the diameter.

**Remark:**

- \(S\) is semi-simple algebra, \(T \subseteq S\), so \(V\) is direct sum of irreducible \(T\)-modules.
- \(V\) is a faithful \(T\)-module, so every irreducible \(T\)-module appears in the direct sum decomposition of \(V\).
**P-structure and Q-structure of J(N,D)**

\[ \chi_i | H \]

\[ \chi^{(1)}_{\alpha} \chi^{(2)}_{\beta} \]

\[ \pi^{(1)}_i \pi^{(2)}_i \]

\[ V_0^*, V_1^*, V_\mu^*, V_i^*, V_{\mu+d}^*, V_D^* \]

\[ V_0, V_1, V_\mu, V_i, V_{\mu+d}, V_D \]
In case of $J(N,D)$,

\[\{\text{Isomorphism classes of irreducible } S\text{-modules}\}\]

\[\Updownarrow \quad \text{Group representation theory}\]

\[\{ (\alpha, \beta) \in \Lambda \}\]

\[\Updownarrow \quad \text{Mapping in 2nd key theorem for } N \neq 2D\]

\[\{ (\nu, \mu, d) \in \Delta \}\]

\[\Updownarrow \quad \text{Remark} + \text{ Terwilliger theorem}\]

\[\{\text{Isomorphism classes of irreducible } T\text{-modules}\}\]
Explanation of $T \subset S$ for $N = 2D$

**Main Theorem**

For Johnson scheme $J(N,D) (2D \leq N)$,
- if $N \neq 2D$, then $T = S$;
- if $N = 2D$, then $T \subset S$.

**Remark:** For $N = 2D$,

$$X = \binom{\Omega}{D}, \quad G = \text{Sym}(\Omega) \subset \text{Sym}(X),$$

$$\tilde{G} := \text{Aut} J(N,D) = G < \sigma >,$$

where $\sigma : X \rightarrow X, x \mapsto \bar{x} = X - x,$

$$\tilde{H} := \tilde{G}_{x_0}.$$

$\blacksquare \quad T = \text{Hom}_{\tilde{H}} (V, V).$
References


Thanks for your attention!