Flows of Signed Graphs: From Modular Flows to Integer Valued Flows

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Joint work with Jian Cheng, You Lu and Cun-Quan Zhang
- $D$: an orientation of $G$. 

$E^-(v)$: the set of all oriented-in edges at $v$. 

$E^+(v)$: the set of all oriented-out edges at $v$. 

$f: E(G) \rightarrow \mathbb{Z}$. 

The ordered pair $(D, f)$ is called an integer flow of $G$ if for every $v \in V(G)$, 

\[ \sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e) \] 

The support of $(D, f)$, $\text{supp}(f) = \{ e \in E(G) : f(e) \neq 0 \}$. 

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The support of $(D, f)$, $\text{supp}(f) = \{ e \in E(G) : f(e) \neq 0 \}$. 
Nowhere-zero $k$-flows

- $G$ admits a $k$-flow if $G$ has a flow $(D, f)$ with $|f(e)| \leq k - 1$ for each edge $e$. 

A $k$-flow $(D, f)$ is nowhere-zero if $f(e) \neq 0$ for every edge.
Nowhere-zero $k$-flows

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- A $k$-flow $(D, f)$ is **nowhere-zero** if $f(e) \neq 0$ for every edge.
An example

Nowhere zero 4−flow
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- If $G$ admits a nowhere-zero $k$-flow, then $G$ admits a nowhere-zero $h$-flow for any integer $h \geq k$.  

Bridges: It is easy to see that if a graph has a bridge, then the flow of the bridge must be zero and thus it does not admit a nowhere-zero flow.
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A graph \( G \) admits a nowhere-zero 2-flow if and only if \( G \) is Eulerian.
The concept of integer flow was originally introduced by Tutte in 1949 as a generalization of map coloring problems.
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**Theorem (Tutte)**

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Let $G$ be a bridgeless graph embedded in an orientable surface. $G$ is face-$k$-coloring $\Rightarrow G$ admits a nowhere-zero $k$-flow.
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The integer flow problem of ordinary graphs is a dual problem of vertex coloring of graphs embedded in orientable surfaces.
Theorem (Seymour)

Every bridgeless graph admits a nowhere-zero 6-flow.
Tutte’s flow conjectures

Conjecture (Tutte’s flow conjectures)

3-flow Conjecture: Every 4-edge connected graph admits a nowhere-zero 3-flow.

4-flow Conjecture: Every bridgeless Petersen minor-free graph admits a nowhere-zero 4-flow.

5-flow Conjecture: Every bridgeless graph admits a nowhere-zero 5-flow.

Thomassen proved that every 8-edge connected graphs admits a nowhere-zero 3-flow.
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Modular Flows-A powerful tool

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and \( D \) be an orientation of \( G \).

The ordered pair \((D, f)\) is called a nowhere-zero modular \( k\)-flow of \( G \) if \( f(e) \neq 0 \) for every edge and for every \( v \in V(G) \),

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**Theorem (Tutte)**

A graph admits a nowhere-zero integer $k$-flow if and only if it admits a nowhere-zero modular $k$-flow.
• $\mathbb{Z}_2 = \{0, 1\}$ and $-1 = 1$. 

\[ Z_3 = \{0, 1, -1\} \] 

For a $\mathbb{Z}_3$-flow $(D, f)$, we may assume $f(e) = 1$.

A graph admits a nowhere-zero 3-flow if and only if it has an orientation $D$ such that for each vertex $v$ 

\[ \sum_{e \in E^+} (v) f(e) = |E^+ (v)| \equiv |E^-(v)| = \sum_{e \in E^- (v)} f(e) \pmod{3} \] 

A cubic graph admits a nowhere-zero 3-flow if and only if it is bipartite.
$\mathbb{Z}_2 = \{0, 1\}$ and $-1 \equiv 1$.

A connected graph admits a nowhere-zero 2-flow if and only if it is Eulerian.
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A cubic graph admits a nowhere-zero 3-flow if and only if it is bipartite.
Let $k$ and $d$ be two positive integers. An integer valued (or modular) circular $\frac{k}{d}$-flow of $G$ is an integer valued (or modular) flow $f$ such that $d \leq |f(e)| \leq k - d$ for every edge $e$.

Circular flow is a refinement of integer flow.
Circular flows and Modular orientations

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An orientation \( D \) of \( G \) is a modular \((2p + 1)\)-orientation if \(|E^+(v)| - |E^-(v)| \equiv 0 \pmod{2p + 1} \) for each vertex \( v \).
Circular flows and Modular orientations

Definition

Let $k$ and $d$ be two positive integers. An integer valued (or modular) circular $\frac{k}{d}$-flow of $G$ is an integer valued (or modular) flow $f$ such that $d \leq |f(e)| \leq k - d$ for every edge $e$.

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Definition

An orientation $D$ of $G$ is a modular $(2p + 1)$-orientation if $|E^+(v)| - |E^-(v)| \equiv 0 \pmod{2p+1}$ for each vertex $v$.

- A graph admits a nowhere-zero 3-flow if and only if it has a modular 3-orientation (if and only if it has a nowhere-zero modular 3-flow).
Theorem (Jaeger)

Let $G$ be a graph. Then the following statements are equivalent:

(A) $G$ admits a modular $(2p + 1)$-orientation.

(B) $G$ admits a modular circular $(2 + \frac{1}{p})$-flow.

(C) $G$ admits an integer-valued circular $(2 + \frac{1}{p})$-flow.
A signed graph is a graph $G$ with a mapping $\sigma : E(G) \rightarrow \{1, -1\}$. 

Signed graphs were first introduced by Harary to handle a problem in social psychology (Cartwright and Harary, 1956).
A signed graph is a graph $G$ with a mapping $\sigma : E(G) \rightarrow \{1, -1\}$. An edge $e \in E(G)$ is positive if $\sigma(e) = 1$ and negative if $\sigma(e) = -1$. Signed graphs were first introduced by Harary to handle a problem in social psychology (Cartwright and Harary, 1956).
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Signed graphs– Switch operation

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- Switching a vertex means negating the signs of all the edges incident to that vertex.
- Switching a set of vertices means negating all the edges that have one end in that set and one end in the complementary set.
- Switching a series of vertices, once each, is the same as switching the whole set at once.
- Two graphs are equivalent under the switching if one can be obtained from the other by switching at a set of vertices.
Signed graphs– Switch operation

Switching at $v$

Switching at $v$

$\Rightarrow$

$\Rightarrow$

$\begin{array}{c}
- \\
\downarrow v \\
+ \\
\end{array} 
\Rightarrow 

\begin{array}{c}
+ \\
\downarrow v \\
- \\
\end{array} 

Orientation of signed graphs

\[ e \in E^-(u) \cap E^+(v) \]
\[ e \in E^+(u) \cap E^-(v) \]

positive edges

negative edges

Figure: Orientation of positive and negative edges
Orientation of signed graphs
Switch operation on orientation
Switch operation on orientation
Switch operation on orientation
Let $G$ be a signed graph and $\tau$ be an orientation of $G$. An ordered pair $(\tau, f)$ is called an integer-valued $k$-flow of $G$ if for every $v \in V(G)$,

$$|f(e)| \leq k$$

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$
Examples of integer flows

nowhere-zero 3-flow

nowhere-zero 4-flow
Examples of integer flow (Cont.)

Infinite family of signed graphs with flow number 6—Schubert & Steffen

Infinite family of signed graphs with flow number 6—Schubert & Steffen
Conjecture (Bouchet, 1983)

Every flow-admissible signed graph admits a nowhere-zero 6-flow.
Let $G$ be a signed graph and $\tau$ be an orientation of $G$. An ordered pair $(\tau, f)$ is called a modular $k$-flow if for every $v \in V(G)$,

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Admitting a nowhere-zero (integer valued or modular) $k$-flow is an invariant under switching operations.
Example

Figure: Flow in Signed $C_5$
Example
An observation

If a signed graph $G$ has only one negative edge, then it does not admits a nowhere-zero $k$-flow for any integer $k$. 
Bridges are allowed
What signed graphs have a nowhere-zero integer flow?

Theorem (Bouchet, 1983, JCTB)

A connected signed graph $G$ admits a nowhere-zero integer flow if and only neither of the following holds:

1. $G$ is switching equivalent to a signed graph with only one negative edge.
2. $G$ has a cut-edge $e$ for which $G - e$ has a balanced component.
Recall—Tutte’s Theorem and Jaeger’s Theorem

Theorem (Tutte)

A graph admits a nowhere-zero integer $k$-flow if and only if it admits a nowhere-zero modular $k$-flow.

Theorem (Jaeger)

Let $G$ be a graph. Then the following statements are equivalent:

(A) $G$ admits a modular $(2p + 1)$-orientation.

(B) $G$ admits a modular circular $(2 + \frac{1}{p})$-flow.

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Recall–Tutte’s Theorem and Jaeger’s Theorem

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Tutte’s Theorem and Jaeger’s Theorem both fail for signed graphs.

For Jaeger’s Theorem, A and B remain equivalent but B and C are not equivalent for signed graphs.
Every Eulerian signed graph admits a nowhere-zero $\mathbb{Z}_2$-flow.
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**Theorem (Xu and Zhang)**

A connected signed graph admits a nowhere-zero integer 2-flow if and only if it is Eulerian and has even number of negative edges.
Another example

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\((G, \sigma)\) has a modular 3-orientation and admits a nowhere-zero modular 3-flow with all edges assigned with 1, but no integer-valued nowhere-zero 3-flow.
How to convert from Modular flows to Integer flows for signed graphs?
Theorem (Xu and Zhang)

Let \((G, \sigma)\) be a signed graph and \(\phi_1\) be a \(\mathbb{Z}_2\)-flow of \((G, \sigma)\) such that \(\text{supp}(\phi_1)\) contains an even number of negative edges. If \(\text{supp}(\phi_1)\) is connected, then \((G, \sigma)\) admits an integer 2-flow \(f_1\) such that \(\text{supp}(\phi_1) = \text{supp}(f_1)\).
Theorem (Xu and Zhang)

Let \((G, \sigma)\) be a signed graph and \(\phi_1\) be a \(Z_2\)-flow of \((G, \sigma)\) such that supp\((\phi_1)\) contains an even number of negative edges. If supp\((\phi_1)\) is connected, then \((G, \sigma)\) admits an integer 2-flow \(f_1\) such that supp\((\phi_1)\) = supp\((f_1)\).

Theorem (Xu and Zhang)

Let \((G, \sigma)\) be a signed graph and \(\phi_2\) be a \(Z_3\)-flow of \((G, \sigma)\). If supp\((\phi_2)\) is bridgeless, then \((G, \sigma)\) admits an integer 3-flow \(f_2\) such that supp\((\phi_2)\) = supp\((f_2)\).
Theorem (Chen, Lu, Luo and Zhang)

Let $(G, \sigma)$ be a connected signed graph and $\phi_1$ be a $\mathbb{Z}_2$-flow of $(G, \sigma)$ such that supp$(\phi_1)$ contains an even number of negative edges. Then $(G, \sigma)$ admits an integer 3-flow $f_1$ such that

$$\text{supp}(\phi_1) = \{ e \in E(G) : f_1(e) = \pm 1 \}.$$
Theorem (Chen, Lu, Luo and Zhang)

Let \((G, \sigma)\) be a connected signed graph and \(\phi_1\) be a \(\mathbb{Z}_2\)-flow of \((G, \sigma)\) such that \(\text{supp}(\phi_1)\) contains an even number of negative edges. Then \((G, \sigma)\) admits an integer 3-flow \(f_1\) such that \(\text{supp}(\phi_1) = \{e \in E(G) : f_1(e) = \pm 1\}\).

Theorem (Chen, Lu, Luo and Zhang)

Let \((G, \sigma)\) be a bridgeless signed graph and \(\phi_2\) be a \(\mathbb{Z}_3\)-flow of \((G, \sigma)\). Then \((G, \sigma)\) admits an integer 4-flow \(f_2\) such that \(\text{supp}(\phi_2) \subseteq \{e \in E(G) : f_2(e) = \pm 1, \pm 2\}\).
[(B)] \((G, \sigma)\) admits a modular circular \((2 + \frac{1}{p})\)-flow.

[(C)] \((G, \sigma)\) admits an integer-valued circular \((2 + \frac{1}{p})\)-flow.

**Theorem**

Let \((G, \sigma)\) be a signed graph. Then (B) and (C) are equivalent if

1. [Xu and Zhang] \(p = 1, \text{ and, } (G, \sigma)\) is cubic and contains a perfect matching;
2. [Schubert and Steffen] \((G, \sigma)\) is \((2p + 1)\)-regular and contains an \(\sigma\)-factor;
3. [Zhu] \((G, \sigma)\) is \((12p - 1)\)-edge-connected with negativeness even or at least \((2p + 1)\).
[(B)] \((G, \sigma)\) admits a modular circular \((2 + \frac{1}{p})\)-flow.

[(C)] \((G, \sigma)\) admits an integer-valued circular \((2 + \frac{1}{p})\)-flow.

**Theorem (Chen, Lu, Luo and Zhang)**

\((B)\) and \((C)\) are equivalent for signed graphs with odd-edge-connectivity at least \((2p + 1)\). That is, if a signed graph \((G, \sigma)\) is odd-\((2p + 1)\)-connected, then it admits a modular circular \((2 + \frac{1}{p})\)-flow if and only if it admits an integer-valued circular \((2 + \frac{1}{p})\)-flow.
An applications of our result to the integer-valued flow
Conjecture (Bouchet, 1983)

Every signed graph admitting a nowhere-zero integer flow admits a nowhere-zero integer 6-flow.

Theorem (DeVos)

Every signed graph admitting a nowhere-zero integer flow admits a nowhere-zero integer 12-flow.
Conjecture (Bouchet, 1983)

Every signed graph admitting a nowhere-zero integer flow admits a nowhere-zero integer 6-flow.

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Known results

**Theorem**

Let $(G, \sigma)$ be a signed graph admitting an NZF. Then

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Let \((G, \sigma)\) be a signed graph admitting an NZF. Then

1. (Zýka) \((G, \sigma)\) admits a 30-NZF.
2. (Lu, Luo, Zhang) If \((G, \sigma)\) contains no edge-disjoint unbalanced circuits, then \(G\) admits a nowhere-zero 6-flow.
3. (Rollov´a et al.) \((G, \sigma)\) admits an 8-NZF if \(G\) is 2-edge-connected and \(|E_N(G, \sigma)| = 2\).
4. (Raspaud and Zhu) \((G, \sigma)\) admits a 4-NZF if \(G\) is 4-edge-connected.
5. (Wu et al.) \((G, \sigma)\) admits a 3-NZF if \(G\) is 8-edge-connected.
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### Known results

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2. \((Lu, Luo, Zhang)\) If \((G, \sigma)\) contains no edge-disjoint unbalanced circuits, then \(G\) admits a nowhere-zero 6-flow.
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4. \((Raspaud and Zhu)\) \((G, \sigma)\) admits a 4-NZF if \(G\) is 4-edge-connected.
5. \((Wu et al.)\) \((G, \sigma)\) admits a 3-NZF if \(G\) is 8-edge-connected.
Known results

**Theorem**

Let $(G, \sigma)$ be a signed graph admitting an NZF. Then

1. (Zýka) $(G, \sigma)$ admits a 30-NZF.
2. (Lu, Luo, Zhang) If $(G, \sigma)$ contains no edge-disjoint unbalanced circuits, then $G$ admits a nowhere-zero 6-flow.
3. (Rollová et al.) $(G, \sigma)$ admits an 8-NZF if $G$ is 2-edge-connected and $|E_N(G, \sigma)| = 2$.
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As an application of our results on converting the modular flows into integer-valued flows, we prove the following result.

**Theorem (Cheng, Lu, Luo, Zhang)**

*Every bridgeless signed graph admitting an NZF admits a 11-NZF.*
Theorem

Let \((G, \sigma)\) be a connected signed graph and \(\phi_1\) be a \(\mathbb{Z}_2\)-flow of \((G, \sigma)\) such that \(\text{supp}(\phi_1)\) contains an even number of negative edges. Then \((G, \sigma)\) admits an integer 3-flow \(f_1\) such that 
\[\text{supp}(\phi_1) = \{e \in E(G) : f_1(e) = \pm 1\} .\]

Theorem

Let \((G, \sigma)\) be a bridgeless signed graph and \(\phi_2\) be a \(\mathbb{Z}_3\)-flow of \((G, \sigma)\). Then \((G, \sigma)\) admits an integer 4-flow \(f_2\) such that 
\[\text{supp}(\phi_2) \subseteq \{e \in E(G) : f_2(e) = \pm 1, \pm 2\} .\]
Proof of the 11-Flow Theorem

Theorem

Every bridgeless flow-admissible signed graph has a nowhere-zero 11-flow.
Proof of the 11-Flow Theorem

**Theorem**

*Every bridgeless flow-admissible signed graph has a nowhere-zero 11-flow.*

We want to show that \((G, \sigma)\) has a flow \(\phi\) such that \(0 < |\phi(e)| \leq 10\) for each edge.
Proof of the 11-Flow Theorem

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**Lemma (Devos)**

*Every flow-admissible signed graph admits a balanced \(\mathbb{Z}_2 \times \mathbb{Z}_3\)-flow.*
Proof of the 11-Flow Theorem

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**Lemma (Devos)**

*Every flow-admissible signed graph admits a balanced \(\mathbb{Z}_2 \times \mathbb{Z}_3\)-flow.***

Let \((G, \sigma)\) be a signed graph and \(\phi = \phi_1 \times \phi_2 : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_3\) be a flow of \((G, \sigma)\), where \(\phi_1\) is a \(\mathbb{Z}_2\)-flow of \((G, \sigma)\) and \(\phi_2\) is a \(\mathbb{Z}_3\)-flow of \((G, \sigma)\). \(\phi\) is called *balanced* if \(\text{supp}(\phi_1)\) contains an even number of negative edges.
Proof of the 11-Flow Theorem

- By Devos’s Lemma, \((G, \sigma)\) admits a balanced \(\mathbb{Z}_2 \times \mathbb{Z}_3\)-flow \(\phi = \phi_1 \times \phi_2\) with \(\text{supp}(\phi_1) \cup \text{supp}(\phi_2) = E(G)\).
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- A balanced \(\mathbb{Z}_2\)-flow \(\mapsto\) an integer 3-flow \(f_1\).
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Then \(e \not\in \text{supp}(\phi_1) \cup \text{supp}(\phi_2)\).

\[\text{supp}(\phi_1) \cup \text{supp}(\phi_2) = E(G)\]

Clearly, such an edge \(e\) does not exist.
Applying the weak 2-linage theorem due to Thomassen and Seymour independently, we give a characterization of signed graphs.

Theorem (Lu, Luo, Zhang)

Let \((G, \sigma)\) be a 2-connected signed graph with negativeness \(\epsilon = |E_N| = k \geq 2\), where \(E_N = \{x_1y_1, x_2y_2, \ldots, x_ky_k\}\) is the set of negative edges of \((G, \sigma)\). Then the following are equivalent:

1. \((G, \sigma)\) contains no edge-disjoint unbalanced circuits.
2. There is a permutation \(\pi\) on \([1, k]\) and \(G - E_N(G)\) is contractible to the 2k-circuit \(\hat{z}_1\hat{z}_2\ldots\hat{z}_{2k}\hat{z}_1\) or to a graph obtained from a 2-connected plane cubic graph by selecting a facial circuit and inserting the 2k vertices \(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_{2k}\) in that cyclic order on edges of the circuit, where \(\{z_i, z_{k+i}\} = \{x_{\pi(i)}, y_{\pi(i)}\}\) for \(i \in [1, k]\).
An example

An illustration of the contracted signed graph with $|E_N| = 5$
Integer flows of signed graphs without edge disjoint unbalanced circuits

Applying the characterization, we have

**Theorem (Lu, Luo, Zhang)**

*Every flow-admissible signed graph without edge disjoint unbalanced circuits admits a 6-NZF.*
Thank you very much.