Permutations, X-rays, Tournaments, Partial Latin Squares, Transversals, and Skolem Sequences

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Joint work with Eliseu Fritscher
Permutations and X-rays

Tournaments

Hankel and Toeplitz X-rays

References
Permutations

\(\sigma: a\) permutation \((j_1, j_2, \ldots, j_n)\) of \(\{1, 2, \ldots, n\}\)

**Permutation matrix** associated with \(\sigma\): \(n \times n\) (0, 1)-matrix

\[
P = [p_{ij}] \text{ where } p_{1j_1} = p_{2j_2} = \cdots = p_{nj_n} = 1.
\]

**Example:** \(n = 4\) and \(\sigma = (2, 4, 1, 3)\):

\[
P_\sigma = \begin{bmatrix}
1 & & & \\
& 1 & & \\
& & 1 & \\
& & & 1
\end{bmatrix}.
\]

(0s are suppressed).
Permutations

\( \sigma: \) a permutation \((j_1, j_2, \ldots, j_n)\) of \(\{1, 2, \ldots, n\}\)

**Permutation matrix** associated with \(\sigma\): \(n \times n\) \((0, 1)\)-matrix

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& 1 & & \\
& & 1 & \\
& & & 1
\end{bmatrix}.
\]

(0s are suppressed).
Toeplitz Matrices

**Toeplitz matrix:** Constant along diagonals parallel to the main diagonal (the **Toeplitz diagonal**):

\[
\begin{array}{cccccc}
5 & 6 & 7 & 8 & 9 \\
4 & 5 & 6 & 7 & 8 & \big(9\big) \\
3 & 4 & 5 & 6 & 7 & \big(8\big) \\
2 & 3 & 4 & 5 & 6 & \big(7\big) \\
1 & 2 & 3 & 4 & 5 & \big(6\big) \\
\end{array}
\]

\(n = 5\)

In general, there are \((2n - 1)\) Toeplitz diagonals. The \(k\text{th} \ \text{Toeplitz diagonal} \) is given by:

\[
\{(i, j) : i - j = n - k\} \ (k = 1, 2, \ldots, 2n - 1).
\]

\((k = n - (i - j))\)
Hankel Matrices

**Hankel matrix:** Constant along diagonals parallel to the antidiagonal (the **Hankel diagonal**):

\[
\begin{array}{cccccc}
 & 1 & 2 & 3 & 4 & 5 \\
(1) & 2 & 3 & 4 & 5 & 6 \\
(2) & 3 & 4 & 5 & 6 & 7 \\
(3) & 4 & 5 & 6 & 7 & 8 \\
(4) & 5 & 6 & 7 & 8 & 9 \\
(5) & (6) & (7) & (8) & (9)
\end{array}
\]

\((n = 5)\)

In general, there are \((2n - 1)\) Hankel diagonals. The \(k\)th Hankel diagonal is given by:

\[
\{(i, j) : i + j = k + 1\} \ (k = 1, 2, \ldots, 2n - 1).
\]

\((k = i + j - 1)\)
Toeplitz X-ray

\[ P = [p_{ij}] \text{ an } n \times n \text{ permutation matrix. It’s Toeplitz X-ray is} \]

\[ t(P) = (t_1, t_2, \ldots, t_{2n-1}) \text{ where} \]

\[ t_k = \sum_{i-j=n-k} p_{ij} \text{ (sum of entries on the } k\text{th Toeplitz diagonal).} \]

Example: \[ P = I_4 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \]

\[ t(P) = (0, 0, 0, 4, 0, 0, 0) \]
Toeplitz X-ray

\( P = [p_{ij}] \) an \( n \times n \) permutation matrix. It’s **Toeplitz X-ray** is

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**Example:** \( P = I_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \)

\[ t(P) = (0, 0, 0, 4, 0, 0, 0) \]
Hankel X-ray

\[ P = [p_{ij}] \] an \( n \times n \) permutation matrix. It’s **Hankel X-ray** is

\[ h(P) = (h_1, h_2, \ldots, h_{2n-1}) \]

where

\[ h_k = \sum_{i+j=k+1} p_{ij} \] (sum of entries on the \( k \)th Hankel diagonal).

**Example:** \( P = I_4 = \)

\[
\begin{array}{ccc}
1 & & \\
& 1 & \\
& & 1 \\
\end{array}
\]

\[ h(P) = (1, 0, 1, 0, 1, 0, 1) \]
Hankel X-ray

$P = [p_{ij}]$ an $n \times n$ permutation matrix. It’s **Hankel X-ray** is

$$h(P) = (h_1, h_2, \ldots, h_{2n-1})$$

where

$$h_k = \sum_{i+j=k+1} p_{ij} \quad \text{(sum of entries on the } k\text{th Hankel diagonal)}.$$

**Example:** $P = I_4 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

$$h(P) = (1, 0, 1, 0, 1, 0, 1)$$
Hankel X-ray $\leftrightarrow$ Toeplitz X-ray

By reversing the order of the columns, the Toeplitz and Hankel X-rays are interchanged:

**Example:** $I_4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\leftrightarrow$ $L_4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = L_4$

$t(I_4) = (0, 0, 0, 4, 0, 0, 0) = h(L_4)$

$h(I_4) = (1, 0, 1, 0, 1, 0, 1) = t(L_4)$

Determining whether or not a sequence of $2n - 1$ nonnegative integers is the Hankel (resp., Toeplitz) X-ray of a permutation matrix is known to be an NP-complete problem.
Hankel X-ray $\leftrightarrow$ Toeplitz X-ray

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Hankel X-ray ↔ Toeplitz X-ray

By reversing the order of the columns, the Toeplitz and Hankel X-rays are interchanged:

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Determining whether or not a sequence of \( 2n - 1 \) nonnegative integers is the Hankel (resp., Toeplitz) X-ray of a permutation matrix is known to be an NP-complete problem.
More Examples

\[ P = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} \]

\[ t(P) = (0, 1, 0, 1, 1, 1, 0, 1, 0) = h(P) \]

Since both X-rays of \( P \) are binary, \( P \) is a solution of the \( n \)-Queens problem for \( n = 5 \).

\[ \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix} \] is not a solution of the 3-Queens problem as its Hankel X-ray is \((0, 2, 0, 0, 1)\) although its Toeplitz X-ray is \((0, 1, 1, 1, 0)\).
More Examples

\[
P = \begin{bmatrix}
1 & & & & 1 \\
& 1 & & & \\
& & 1 & & \\
& & & 1 & \\
1 & & & & \\
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Tournaments

A tournament $T$ of order $n$ is an orientation of the complete graph $K_n$ of order $n$: For each unordered pair $\{p, q\}$ of distinct integers from $\{1, 2, \ldots, n\}$ one chooses one of $(p, q)$ or $(q, p)$. A tournament has an $n \times n$ adjacency matrix $T = [t_{ij}]$ where $t_{ii} = 0$ for all $i$ and $t_{ij} + t_{ji} = 1$ for all $i \neq j$. Thus

$$T + T^t = J_n - I_n$$

where $J_n$ is the $n \times n$ matrix of all 1s.

Example: $T = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (combinatorially skew symmetric)

The score vector of a tournament $T$ is the vector $R = (r_1, r_2, \ldots, r_n)$ of row sums of $T$. In the example, $R = (2, 1, 2, 1)$. One can, and we do, assume that $R$ is nondecreasing: $r_1 \leq r_2 \leq \cdots \leq r_n$. 

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Landau’s Theorem on Score Vectors of Tournaments

**Theorem:** Let \( R = (r_1, r_2, \ldots, r_n) \) be a vector of nondecreasing, nonnegative integers. Then \( R \) is the score vector of a tournament if and only if

\[
\sum_{i=1}^{k} r_i \geq \binom{k}{2} \quad (k = 1, 2, \ldots, n),
\]

with equality for \( k = n \).

(These conditions are obviously necessary.)

These conditions have an equivalent reformulation.
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Reformulation of Landau’s Theorem

Let $R = (r_1, r_2, \ldots, r_n)$ with $r_1 \leq r_2 \leq \cdots \leq r_n$ satisfy Landau’s inequalities, and set

$$X = (x_1, x_2, \ldots, x_n) = (r_1, r_2, \ldots, r_n) + (1, 2, \ldots, n).$$

Then $1 \leq x_1 < x_2 < \cdots < x_n \leq 2n - 1$ and

$$\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} r_i + \sum_{i=1}^{k} i \geq \binom{k}{2} + \binom{k+1}{2} = k^2,$$

with equality for $k = n$. And this is reversible. $X$ is the adjusted score vector of a tournament.
Reformulation of Landau’s Theorem

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Then \( 1 \leq x_1 < x_2 < \cdots < x_n \leq 2n - 1 \) and

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\]

with equality for \( k = n \). And this is reversible. \( X \) is the adjusted score vector of a tournament.
Number of Score Vectors of Tournaments of Order $n$

The number $s_n$ of possible score vectors of tournaments of order $n$ equals the number of positive integral solutions of the system of linear relations:

$$\begin{align*}
1 \leq x_1 < x_2 < \cdots < x_n \leq 2n - 1 \\
\sum_{i=1}^{k} x_i \geq k^2 \quad (k = 1, 2, \ldots, n \text{ with equality for } k = n.)
\end{align*}$$
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\end{cases}$$

??? Permutations, X-rays, tournaments, score vectors ???
X-rays of Permutations and Score Vectors of Tournaments

\( P = [p_{ij}] \) and \( n \times n \) permutation matrix. Its Toeplitz and Hankel X-rays \( t(P) \) and \( h(P) \) have at most \( n \) nonzero terms, with exactly \( n \) nonzero terms if and only if the X-rays are binary.

Let the **Toeplitz characteristic** of \( P \) be defined as:

\[ \chi_t(P) \text{ is the vector of positions in } t(P) \text{ which are nonzero, taken in non-decreasing order, where we repeat a position a number of times according to its value: } \chi_t(P) = (l_1, l_2, \ldots, l_n) \text{ where } 1 \leq l_1 \leq l_2 \leq \cdots \leq l_n. \]

The \( l_i \) are distinct if and only if \( t(P) \) is binary.

Similarly, we have the **Hankel characteristic**:

\[ \chi_h(P) = (l'_1, l'_2, \ldots, l'_n) \text{ where } 1 \leq l'_1 \leq l'_2 \leq \cdots \leq l'_n \text{ with similar properties.} \]
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where $1 \leq l'_1 \leq l'_2 \leq \cdots \leq l'_n$ with similar properties.
Examples

- First note that if \( \sigma = (j_1, j_2, \ldots, j_n) \), then
  \[
  \chi_t(P_\sigma) = (i_j - j + n : 1 \leq j \leq n) \quad \text{&} \quad \chi_h(P_\sigma) = (i_j + j - 1 : 1 \leq j \leq n),
  \]
  rearranged to be in nondecreasing order.

- \( \chi_t(I_n) = (n, n, \ldots, n) \) and \( \chi_h(I_n) = (1, 3, 5, \ldots, 2n - 1) \).

- If \( P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \), then \( \chi_t(P) = (2, 3, 6, 7, 7) \) and \( \chi_h(P) = (2, 4, 5, 7, 7) \).
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- First note that if $\sigma = (j_1, j_2, \ldots, j_n)$, then
  
  $\chi_t(P_\sigma) = (i_j - j + n : 1 \leq j \leq n)$ & $\chi_h(P_\sigma) = (i_j + j - 1 : 1 \leq j \leq n)$, rearranged to be in nondecreasing order.

- $\chi_t(I_n) = (n, n, \ldots, n)$ and $\chi_h(I_n) = (1, 3, 5, \ldots, 2n - 1)$.

- If $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then $\chi_t(P) = (2, 3, 6, 7, 7)$ and $\chi_h(P) = (2, 4, 5, 7, 7)$. 
Examples

First note that if $\sigma = (j_1, j_2, \ldots, j_n)$, then

$$\chi_t(P_{\sigma}) = (i_j - j + n : 1 \leq j \leq n) \text{ and } \chi_h(P_{\sigma}) = (i_j + j - 1 : 1 \leq j \leq n),$$

rearranged to be in nondecreasing order.

- $\chi_t(I_n) = (n, n, \ldots, n)$ and $\chi_h(I_n) = (1, 3, 5, \ldots, 2n - 1)$.

- If $P = \begin{bmatrix} 1 & & & \cr & 1 & & \cr & & 1 & \cr 1 & & & \end{bmatrix}$, then $\chi_t(P) = (2, 3, 6, 7, 7)$ and $\chi_h(P) = (2, 4, 5, 7, 7)$. 
Binary X-rays and Tournaments

Let $\sigma = (i_1, i_2, \ldots, i_n)$ be a permutation such that $P_\sigma$ has a **binary Hankel X-ray** (could use Toeplitz X-ray instead). Then

$$
\chi_h(P_\sigma) = (i_1 + 0, \ldots, i_j + j - 1, \ldots, i_n + n - 1)
= (i_1, i_2, i_3, \ldots, i_n) + (0, 1, 2, \ldots, n - 1),
$$

which are $n$ distinct, not necessarily monotone, integers between 1 and $2n - 1$.

The sum of the $k$ smallest of the integers in $\chi_h(P_\sigma)$ is

$$
\geq \sum_{j=1}^k i + \sum_{j=1}^k (j - 1) = \binom{k + 1}{2} + \binom{k}{2} = k^2,
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with equality when $k = n$. Thus $\chi_h(P_\sigma)$ is the adjusted score vector of a tournament of order $n$. 
Binary X-rays and Tournaments

Let $\sigma = (i_1, i_2, \ldots, i_n)$ be a permutation such that $P_\sigma$ has a **binary Hankel X-ray** (could use Toeplitz X-ray instead). Then

$$\chi_h(P_\sigma) = (i_1 + 0, \ldots, i_j + j - 1, \ldots, i_n + n - 1)$$

$$= (i_1, i_2, i_3, \ldots, i_n) + (0, 1, 2, \ldots, n - 1),$$

which are $n$ distinct, not necessarily monotone, integers between 1 and $2n - 1$.

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**Conclusion:**

\[
\{ \text{Binary Hankel X-rays of } n \times n \text{ permutation matrices} \} \downarrow \text{injective} \\
\{ \text{Score vectors of tournaments of order } n \}
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**Conjecture:** (Bebeacua, Mansour, Postnikov, and Severini, 2005)
This map is surjective, that is, the number of binary Hankel X-rays of \( n \times n \) permutation matrices equals the number of score vectors of tournaments of order \( n \).

**Note:** This correspondence is between score vectors of tournaments (not tournaments) and binary Hankel X-rays of permutations (not permutations with binary Hankel X-rays).
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Binary X-rays, Tournaments, and Partial Latin Squares

Let $X = (x_1 < x_2 < \cdots < x_n)$ be an increasing sequence of positive integers with $\sum_{i=1}^{k} x_i \geq k^2$ with equality for $k = n$ (an adjusted score vector of a tournament) and define array

$$L_X = \begin{array}{cccccc}
0 & 1 & \cdots & j-1 & \cdots & n-1 \\
\hline
x_1 & & & & & \\
x_2 & & & & & \\
\vdots & & & & & \\
x_i & & & & b & \\
\vdots & & & & & \\
x_n & & & & & \\
\end{array}$$

where we put the integer $b$ in the $(x_i, j-1)$ position provided that $x_i = b + (j - 1)$, otherwise leave $\emptyset$. A permutation $P$ with Hankel characteristic $X$ exists iff $L_X$ has a transversal.
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Example of a Partial Latin Square for a Permutation

Let $\sigma = (3, 1, 2, 4, 5)$ and $P_\sigma = \begin{bmatrix}
1 & & & & \\
& 1 & & & \\
& & 1 & & \\
& & & 1 & \\
& & & & 1
\end{bmatrix}$. Then $h(P) = (0, 1, 1, 1, 0, 0, 1, 0, 1)$ and $\chi_h(P) = (2, 3, 4, 7, 9)$:

$$2 + 3 + 4 + 7 + 9 = 5^2.$$ The corresponding partial Latin square corresponding to $X = (2, 3, 4, 7, 9)$ with a transversal shaded (there is only one) is

$$L_X = \begin{array}{cccc|c}
0 & 1 & 2 & 3 & 4 \\
\hline
x_1 = 2 & \quad & 2 & 1 & \\
\hline
x_2 = 3 & \quad & 3 & 2 & 1 \\
\hline
x_3 = 4 & \quad & 4 & 3 & 2 & 1 \\
\hline
x_4 = 7 & \quad & & 5 & 4 & 3 \\
\hline
x_5 = 9 & \quad & & & & 5
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Skolem Sequences

**Skolem set:** a sequence of integers $x_1 < x_2 < \cdots < x_n$ such that there is a sequence of length $2n$, a **Skolem sequence**, consisting of two copies of each $x_i$ where the two copies of $x_i$ are $x_i$ apart.

**Example:** $\{2, 3, 5, 6\}$ is a Skolem set with corresponding Skolem sequence $5, 6, 2, 3, 2, 5, 3, 6$.

**Necessary condition for a Skolem set** (a calculation) is that

\[
(*) \sum_{i=k+1}^{n} x_i \leq n^2 - k^2 \quad (k = 0, 1, \ldots, n - 1),
\]

In particular, $\sum_{i=1}^{n} x_i \leq n^2$. If equality holds, then (*) becomes

\[
(**) \sum_{i=1}^{k} x_i \geq k^2 \quad (k = 1, 2, \ldots, n \text{ with equality if } k = n).
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Skolem sets/sequences satisfying (**) are extremal Skolem sets/sequences, e.g. $\{2, 3, 5, 6\}$ and $5, 6, 2, 3, 2, 5, 3, 6$. 
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Conjecture: \( \{x_1 < x_2 < \cdots < x_n\} \) is an extremal Skolem set if and only if

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Recall the Conjecture: The number of binary Hankel X-rays of \( n \times n \) permutation matrices equals the number of score vectors of tournaments of order \( n \).

So IF the two conjectures are true, there is a one-to-one correspondence between binary Hankel X-rays of \( n \times n \) permutation matrices and extremal Skolem sequences (and score vectors of tournaments of order \( n \)).
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$n$-Queens Problem

Equivalent to $n \times n$ permutation matrices with binary Toeplitz and Hankel X-rays. A solution exists for all $n \geq 4$

Powerful Toeplitz Queen: a Queen that can attack another in the same row, column, Toeplitz diagonal, and the symmetrically opposite Toeplitz diagonal (but not in the same Hankel diagonal). In terms of permutation matrices, there must be a 1 on the main diagonal, and exactly one of the Toeplitz diagonals, $i$ and $2n - i$, must contain a 1 for $i = 1, 2, \ldots, n - 1$.

Examples:

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
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1 \\
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Fact: Solution exists if and only if $n \equiv 0, 1 \mod 4$, \ldots
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Let \( l = (l_1, l_2, \ldots, l_n) \) with \( 1 \leq l_1 \leq l_2 \leq \cdots \leq l_n \leq 2n - 1 \) and \( l' = (l'_1, l'_2, \ldots, l'_n) \) with \( 1 \leq l'_1 \leq l'_2 \leq \cdots \leq l'_n \leq 2n - 1 \) be given. When is there an \( n \times n \) permutation matrix \( P \) such that \( \chi_h(P) = l \) and \( \chi_t(P) = l' \)? The same question under the assumption that \( h \) and \( t \) are binary.

Of course, this is an NP-complete problem but nonetheless worth considering.

Necessary Conditions via some calculation

- \((n \text{ odd})\): The number of odd (resp., even) elements of \( l_h \) equals the number of odd (resp., even) elements of \( l_t \).
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\[ \sum_{i=1}^{n} (l_i^2 + l'_i^2) = \frac{n(7n^2 - 1)}{3} \]
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The Toeplitz X-ray $t(P) = (t_1, t_2, \ldots, t_{2n-1})$ of a permutation matrix $P$ is **palindromic** provided that

$$(t_1, t_2, \ldots, t_{n-1}) = (t_{2n-1}, t_{2n-2}, \ldots, t_{n+1}).$$

(Similar definition for palindromic Hankel X-ray.)

**Examples:**

- Any (Toeplitz) symmetric permutation matrix.

$$
\begin{bmatrix}
1 \\
1 & 1 \\
1 \\
1 & & & & & \\
& & & & & \\
& & & & &
\end{bmatrix}
$$

Then $P$ is not symmetric, but $t(P) = 001111100$ is palindromic but $h(P) = 100110110$ is not.
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Four different constructions yield the following:

**Theorem:** For each \( n \geq 4 \) there exists an \( n \times n \) permutation matrix whose Hankel and Toeplitz X-rays are binary and palindromic.

**Example:** A construction that works for \( n \) even and \( 3 \not| n \):

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\begin{array}{cccc}
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Permutations with equal Hankel and Toeplitz X-rays I

Can they exist? Sometimes but not always.

**Theorem:** If \( n \equiv 2 \mod 4 \), then there does not exist an \( n \times n \) permutation matrix whose Hankel X-ray equals its Toeplitz X-ray.

**Theorem:** Let \( n \) be an odd integer. Let \( P \) be an \( n \times n \) permutation matrix with equal palindromic Hankel and Toeplitz X-rays whose Hankel and Toeplitz characteristics equal \( (l_1, l_2, \ldots, l_n) \). Then

\[
12 \sum_{i=1}^{n-1} \frac{l_i}{n+1-i} = (n - 1)n(5n - 1).
\]

In particular, if \( n \equiv 3 \mod 4 \), then the number of odd integers in \( l_1, l_2, \ldots, l_{\frac{n-1}{2}} \) is odd.
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$$12 \sum_{i=1}^{\frac{n-1}{2}} l_i l_{n+1-i} = (n - 1)n(5n - 1).$$

In particular, if $n \equiv 3 \text{ mod } 4$, then the number of odd integers in $l_1, l_2, \ldots, l_{\frac{n-1}{2}}$ is odd.
**Corollary:** There does not exist a $7 \times 7$ permutation matrix with equal palindromic Hankel and Toeplitz X-rays. Such a permutation would have Hankel characteristic $(l_1, l_2, l_3, l_4, l_5, l_6, l_7)$ with one or all of $l_1, l_2, l_3$ odd integers and, by the theorem, with $\sum_{i=1}^{3} l_i l_{8-i} = 119$. It is easy to check that this is not possible.
Some References