Chapter 3 Free Boundary

In this section, we establish a few properties of the free boundary for American Put Option problem. We focus on the technique.

§1 The free boundary

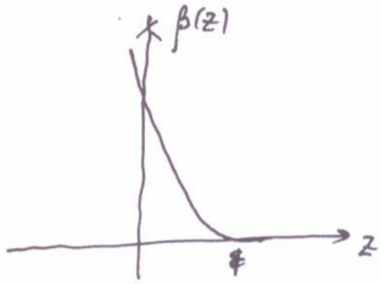
$$\begin{cases} \min\{\mathcal{L}p,\ p - p_0\} = 0 & \text{in } \mathbb{R}^{*} \times (0,\infty) \\ p - p_0 = 0 & \text{on } \mathbb{R}^{*} \times \{0\} \end{cases}$$

$$p_0(x) = \max\{1 - e^x, 0\} = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 - e^x & \text{if } x < 0 \end{cases}$$

$$\mathcal{L}p = p_t - p_{xx} - (k-1)p_x + kp \quad ; \quad k = \frac{2r}{\sigma^2}.$$

Approximation by the penalily method.

$$\begin{cases} \mathcal{L} p_\varepsilon \cancel{\{ - p_\varepsilon - p_0^\varepsilon \} = 0} = \beta_\varepsilon(p_\varepsilon - p_{0\varepsilon}) & \text{in } \mathbb{R} \times (0,\infty) \\ p_\varepsilon - p_0^\varepsilon = 0 & \text{in } \mathbb{R} \times \{0\} \end{cases}$$



$\beta_\varepsilon(z) = \beta\left(\frac{z}{\varepsilon}\right)$

<u>Lemma 1</u> $p_\varepsilon^0 \leq p_\varepsilon \leq 1$.

<u>Proof</u> Define $\mathcal{F}(p) = \mathcal{L}p - \beta_\varepsilon(p - p_0^\varepsilon))$. Then

$$\mathcal{F}(\bar{p})\Big|_{\bar{p} \equiv 1} = k - \beta_\varepsilon(1 - p_0^\varepsilon) \geq k - \beta(0) = 0.$$

Hence, by comparison, $p_\varepsilon \leq 1$ on $(0,\infty)^2$.

$$\mathcal{F}(\underline{p})\Big|_{\underline{p} = p_0^\varepsilon} = \mathcal{L}p_0^\varepsilon - \beta_\varepsilon(0) = \mathcal{L}p_0^\varepsilon - k \leq 0.$$

Hence, by comparison, $p \geq p_\varepsilon^0$. QED.

Lemma 2 $\quad 0 \geq \dfrac{\partial p_\varepsilon}{\partial x} \geq -e^{x+\varepsilon}.$

Proof Set $\zeta = \dfrac{\partial p_\varepsilon}{\partial x}$. Then

$$0 = \frac{\partial}{\partial x}\left(\mathcal{L} p_\varepsilon - \beta_\varepsilon(p_\varepsilon - p^0_\varepsilon)\right) \quad ; \qquad \beta_\varepsilon(z) = \beta\left(\frac{z}{\varepsilon}\right)$$

$$= \mathcal{L}\zeta_\varepsilon - \beta_\varepsilon' \left[\zeta_\varepsilon - p^{0\prime}_\varepsilon(x)\right].$$

(1) Assume that $p_0^{\varepsilon\prime}(x) \leq 0$. Then

$$\begin{cases} \mathcal{L}\zeta_\varepsilon - \zeta_\varepsilon \beta_\varepsilon' = -\beta_\varepsilon' p_0^{\varepsilon\prime}(x) \leq 0 \\ \zeta_\varepsilon|_{t=0} = p_{0\varepsilon}'(x) \leq 0 \end{cases}$$

By maximum principle, $\zeta_\varepsilon \leq 0$ in $\mathbb{R} \times (0, \infty)$

(2) $\quad p_0(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 - e^x & \text{if } x \leq 0 \end{cases}$; $\qquad p_0'(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ -e^x & \text{if } x \leq 0. \end{cases}$

$$p_{0\varepsilon}(x) = \rho_\varepsilon * p_0, \qquad p_{0\varepsilon}' = \rho_\varepsilon * p_0' \geq -e^{x+\varepsilon}.$$

$$\mathcal{L}\zeta - \beta_\varepsilon'[\zeta - p^0_\varepsilon)\Big|_{\zeta = -e^{x+\varepsilon}} = -\beta_\varepsilon'\left[-e^{x+\varepsilon} - p_{0\varepsilon}'(x)\right] \geq 0$$

$$\zeta\Big|_{t=0} \geq$$

$$\mathcal{L}(\zeta_\varepsilon + e^{x+\varepsilon}) - \beta_\varepsilon'\left[\zeta_\varepsilon + e^{x+\varepsilon}\right] = -\beta_\varepsilon'\left[p^{0\prime}_\varepsilon + e^{x+\varepsilon}\right] \geq 0$$

$$\zeta_\varepsilon + e^{x+\varepsilon}\Big|_{t=0} \geq 0$$

Thus by Maximum principle, $\zeta_\varepsilon + e^{x+\varepsilon} \geq 0.$

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Lemma 3 $\frac{\partial p_\varepsilon}{\partial t} \geq 0$

Proof 1) $0 = \frac{\partial}{\partial t}\left[\mathcal{L} p_\varepsilon - \beta_\varepsilon(p_\varepsilon - p_{0\varepsilon})\right]$

$= \mathcal{L}\frac{\partial p_\varepsilon}{\partial t} - \beta_\varepsilon' \frac{\partial p_\varepsilon}{\partial t} -$

2) $\frac{\partial p_\varepsilon}{\partial t}\Big|_{t=0} = L p_{0\varepsilon} + \beta_\varepsilon(0) \cancel{\leftarrow k + L}$

$= p_{0\varepsilon}''(x) + (k-1)\,p_{0\varepsilon}'(x) - k\,p_{0\varepsilon} + k \geq 0.$

Sending $\varepsilon \to 0$ we have the following estimates

Lemma 4 (1) $p_0 \leq p \leq 1$

(2) $0 \geq \frac{\partial p}{\partial x} \geq -e^x.$

(3) $\frac{\partial p}{\partial t} \geq 0$

(4) $\frac{\partial p}{\partial x}(S(t), t) = -e^{S(t)}.$

Lemma 5 As $t \to \infty$,

$p(x,t) \to p_*(x)$

which is solution of

$\max\{-p_*'' - (k-1)p_*' + kp_*,\ p_* - p_0\} = 0$ in $\mathbb{R}$

i.e

$$p_*(x) = \begin{cases} 1 - e^x & \text{if } x \leq \ln\frac{1}{1+k} = x_\infty \\ \frac{k}{1+k}e^{-k(x - x_\infty)} & \text{if } x \geq x_\infty \end{cases}$$

Theorem 1 Define $S(0) = 0$ and

$S(t) = \inf\{x \mid p(x,t) > p_0(x)\}. \quad \forall t > 0$

Then the following holds

(1) $p(x,t) > p_0(x)$ in $(S(t), \infty)$

(2) $p(x,t) = p_0(x)$ in $(-\infty, S(t)]$

(3) If $\cancel{0} < t_1 < t_2$, then $0 > S(t_1) > S(t_2) > S_\infty = \ln\frac{k}{k+1}$

(4) Define $S(0) = 0$. Then $S \in C([0,\infty))$, $\lim_{t\to\infty} S(t) = \ln\frac{k}{k+1}$.

(5) Let $t = \tau(x)$ be the inverse of $x = S(t)$. Then $\tau \in C(\cancel{[}(S_\infty, 0])$ where $S_\infty = \ln\frac{k}{k+1}$