Variational Inequalities and Free Boundary Problems

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This mini-course introduces some basic theory on free boundary problems and its associated variational inequality problems. As an example, an American put problem will be thoroughly investigated. An outline of the course is as follows:

Chapter 1 Variational Inequality

1. Obstacle Problem
   Introduce the problem, the definition of variational solution, and the definition of viscosity solution.
2. Variational inequalities arising from mathematical finance
   Present a brief derivation of the model. A numerical scheme is illustrated.
3. The Penalty Method.
   Introduce the classical penalty method, based on the model problem of the variational inequality arising from the American put option.

Chapter 2 Stefan Problem

1. Stefan problem and variational inequality
   Briefly describe the classical Stefan problem. Attention is focus on the derivation of the Stefan problem from an variational inequality.
2. Well-posedness of the Stefan problem
   Taking the free boundary problem arising from the American put option as an example, the well-posedness theory for the one-space dimensional Stefan problem is fully explained.
3. From Stefan problem back to variational Inequality

Chapter 3 Free Boundary

In this chapter we study the free boundary from American Put option problem.

1. Integral formulations
   For theoretical as well as numerical analysis, integral formulations are derived for the free boundary problem arising from American put option.
2. Convexity of free boundary for Stefan problem
3. Regularity of the free boundary
   Present a standard theory, especially the boot strap argument, for the regularity of the free boundary.

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Exam Problems Given by Professor Xinfu Chen

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1. Let $F(q, p, \theta, x) = \min\{-q, \theta - g(x)\}$ for $(q, p, \theta, x) \in \mathbb{R}^3 \times \bar{\Omega}$ where $\Omega = (-1, 1)$ and $g(x) = 1 - 2|x|$. Show that $u(x) = 1 - |x|$ is a viscosity solution of

$$F\big(u_{xx}(x), u_x(x), u(x), x\big) = 0 \quad \forall\, x \in \Omega.$$

As a first step, sketch the curves $y = u(x)$ and $y = g(x)$ in $[-1, 1] \times [-0.1, 1.1]$.

2. Consider the classical Stefan problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \ Q := \{(x,t) \mid t > 0, x > s(t)\}, \\ u = 0 & \text{in } \ \Sigma := \{(x,t) \mid t > 0, x \leqslant s(t)\}, \\ u(x,0) = f(x) & \forall\, x \geqslant 0, \\ \ell \frac{d}{dt} s(t) = -\frac{\partial}{\partial x} u(s(t)^+, t) & \forall\, t > 0, \\ s(0) = 0, \end{cases} \tag{1}$$

where f is a non-negative smooth function defined on $[0, \infty)$. Derive the zeroth order, first order, and second order compatibility conditions on the initial data f, so that $u \in C^{2\alpha,\alpha}(\bar{Q})$ for $\alpha = 0, \alpha = 1$, and $\alpha = 2$, respectively. Hint: compute $\lim_{t \searrow 0} \left[1, \frac{d}{dt}, \frac{d^2}{dt^2}\right] u(s(t), t)$.

3. Let (s,u) be a solution of (1) where $s \in C^1([0,\infty))$ and $u \in C^{2,1}(\bar{Q})$ and $u > 0$ in Q. Define

$$w(x,t) = \int_0^t u(x,\tau)d\tau \qquad \forall\,(x,t) \in \mathbb{R} \times [0,\infty).$$

Show that w is the solution to the following variational inequality

$$\begin{cases} \min\{w_t - w_{xx} - e_0, w\} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ w = 0 & \text{on } \mathbb{R} \times \{0\}, \end{cases}$$

where e_0, called the initial enthalpy, is the function defined by

$$e_0(x) = f(x)\mathbf{1}_{(0,\infty)}(x) - \ell\mathbf{1}_{(-\infty,0)}(x).$$

Hint: consider three cases: (i) $x \geqslant 0$, (ii)$s(t) < x < 0$ and (iii) $x \leqslant s(t)$.

Solution

1. Proof. (1) Suppose $(x_0, \phi) \in \Omega \times C^2(\bar{\Omega})$ is a pair satisfying $\phi \geqslant u$ in $\bar{\Omega}$ and $\phi(x_0) = u(x_0)$.

   (a) If $x_0 = 0$ then $\phi(0) = u(0) = 1$ so $F(\phi''(0), \phi'(0), \phi(0), 0) = \min\{-\phi''(0), \phi(0) - g(0)\} \leqslant 0$.

   (b) If $x_0 \neq 0$, then $\phi - u$ attains a local minimum at x_0 so $\phi''(x_0) - u''(x_0) \geqslant 0$, which implies that $-\phi''(x_0) \leqslant -u''(x_0) = 0$. Again we have $\min\{-\phi''(0), \phi(0) - g(0)\} \leqslant 0$.

   Thus, in any case we have $F(\phi''(0), \phi'(0), \phi(0), 0) \leqslant 0$. By definition, u is a subsolution.

   (2) Suppose $(x_0, \psi) \in \Omega \times C^2(\bar{\Omega})$ is a pair satisfying $\psi \leqslant u$ in $\bar{\Omega}$ and $\psi(x_0) = u(x_0)$.

   (i) If $x_0 \neq 0$, then $\psi - u$ attains a local maximum at x_0 so $\psi''(x_0) - u''(x_0) \leqslant 0$, which implies that $-\psi''(x_0) \geqslant -u''(x_0) = 0$. As $\psi(x_0) = u(x_0) > g(x_0)$, we have $F(\psi''(0), \psi'(0), \psi(0), 0) = \min\{-\psi''(0), \psi(0) - g(0)\} \geqslant 0$.

   (ii) We claim that $x_0 = 0$ is impossible. In deed, it $x_0 = 0$ we would have

$$\psi'(0) = \lim_{h \searrow 0} \frac{\psi(h) - \psi(0)}{h} \leqslant \lim_{h \searrow 0} \frac{u(h) - u(0)}{h} = -1, \qquad \text{and}$$
$$\psi'(0) = \lim_{h \searrow 0} \frac{\psi(0) - \psi(-h)}{h} \geqslant \lim_{h \searrow 0} \frac{u(0) - u(-h)}{h} = 1,$$

   a contradiction.

   Thus, we have $F(\psi''(x_0), \psi'(x_0), \psi(x_0), x_0) \geqslant 0$. By definition, u is a supersolution.

   In conclusion, u is a viscosity solution. □

2. Solution. We calculate

$$\begin{aligned}
0 &= \lim_{t \searrow 0} u(s(t), t) = u(0, 0) = \lim_{x \searrow 0} u(x, 0) = f(0), \\
0 &= \lim_{t \searrow 0} \ell \frac{d}{dt} u(s(t), t) = \lim_{t \searrow 0} \Big\{ -u_x(s(t), t)^2 + \ell u_t(s(t), t) \Big) \\
&= \lim_{x \searrow s(t), t \searrow 0} \Big\{ -u_x(x, t)^2 + \ell u_{xx}(x, t) \Big) = -f'(0)^2 + \ell f''(0). \\
0 &= \lim_{t \searrow 0} \ell^2 \frac{d^2}{dt^2} u(s(t), t) = \lim_{t \searrow 0} \ell \frac{d}{dt} \Big(-u_x(s(t), t)^2 + \ell u_t(s(t), t) \Big\} \\
&= \lim_{x \searrow s(t), t \searrow 0} \Big\{ [-2u_x u_{xx} + \ell u_{xt}](-u_x) + \ell[-2u_x u_{xt} + \ell u_{tt} \Big\} \\
&= \lim_{x \searrow s(t), t \searrow 0} \Big\{ 2u_x^2 u_{xx} - \ell u_{xxx} u_x - 2\ell u_x u_{xxx} + \ell^2 u_{xxxx} \Big\} \\
&= \ell^2 f''''(0) - 3\ell f'''(0) f'(0) + 2f''(0) f'(0)^2.
\end{aligned}$$

   Thus, the zeroth, first, and second order compatibility conditions are

$$\begin{aligned}
f(0) &= 0, \\
\ell f''(0) &= f'(0)^2, \\
\ell^2 f''''(0) &= 3\ell f'''(0) f'(0) - 2f''(0) f'(0)^2.
\end{aligned}$$

3. Proof. We notice that $u > 0$ in Q implies that $u_x(s(t)^+, t) > 0$ so $s'(t) < 0$ for each $t > 0$.

(i) First consider $(x,t) \in D_1 := [0,\infty) \times (0,\infty)$. We have,

$$w_t - w_{xx} = u(x,t) - \int_0^t u_{xx}(x,\tau)d\tau = f(x) + \int_0^t [u_t - u_{xx}](x,\tau)]d\tau = f(x).$$

Since $u > 0$ in Q and $s(t) < 0$ for $t > 0$, we see that $w(x,t) > 0$. Hence,

$$\min\{w_t - w_{xx} - e_0, w\} = 0 \quad \text{in} \;\; D_1.$$

(ii) Next we consider the case when $(x,t) \in D_2 := \{(y,\tau) \mid \tau > 0, y \in (s(t), 0)\}$. Denote by $t = T(x)$ the inverse function of $x = s(t)$. Then $T'(x) = 1/s'(T(x)) = \ell/u_x(x, T(x))$. Thus,

$$w(x,t) = \int_0^t u(x,\tau)d\tau = \int_{T(x)}^t u(x,\tau)d\tau.$$

Consequently,

$$\begin{aligned} w_t(x,t) &= u(x,t) = \int_{T(x)}^t u_t(x,\tau)d\tau, \qquad w_x(x,t) = \int_{T(x)}^t u_x(x,\tau)d\tau, \\ w_{xx}(x,t) &= \int_{T(x)}^t u_{xx}(x,\tau)d\tau - T'(x)u_x(x,T(x)) = \int_{T(x)}^t u_{xx}(x,\tau)d\tau + \ell. \end{aligned}$$

It then follows that

$$w_t - w_{xx} - e_0 = \int_{T(x)}^t [u_t - u_{xx}](x,\tau)d\tau - e_0 + \ell = 0.$$

Hence,

$$\min\{w_t - w_{xx} - e_0, w\} = 0 \quad \text{in} \;\; D_2.$$

(iii) Next consider the case $(x,t) \in D_3 := \{(y,\tau) \mid \tau > 0, y < s(\tau)\}$. Then we have $w = 0$ in D_3. It follows that $\min\{w_t - w_{xx} - e_0, w\} = \min\{-e_0, 0\} = 0$ in D_3.

(iv) Finally consider the case $x = s(t)$. We find that $w_x(x,t) = 0$ for $x \leqslant s(t)$ and $w_{xx}(s(t)^+, t) = \ell$. Thus, we also have $\min\{w_t - w_{xx} - e_0, w\} = \min\{-e_0, 0\} = 0$.

In conclusion, w is a solution of the variational inequality. $\square$.