Solution

1.1

Problem 1

Let $B_i := B(r_i, \frac{1}{2^i})$, For every N we define:

$$E_N := \sum_{i=1}^{N} B_i$$

Clearly:

$$P(E_N) \leq \sum_{i=1}^{N} P(B_i) = n\omega_n \sum_{i=1}^{N} r_i^{n-1} \quad (1).$$

We have $\chi_{E_N} \to \chi_E$ in $L^1(\mathbb{R}^n)$

Since:

$$\int_{\mathbb{R}^n} |\chi_{E_N} - E|\,dx = |E \setminus E_N| \leq \sum_{N+1}^{\infty} |B_i| = \sum_{N+1}^{\infty} \omega_n r_i^n = \sum_{N+1}^{N} \omega_n \frac{1}{(2^n)^i} < \infty \text{ (geometric series)}.$$

Hence:

$$P(E) \leq \liminf_{N\to\infty} P(E_N) \leq \liminf_{N\to\infty} n\omega_n \sum_{i=1}^{N} r_i^{n-1} \text{ ; by (1)}$$

$$< \infty \text{ ; Since } \sum_{i=1}^{\infty} (r_i)^{n-1} < \infty.$$

We conclude that

$\mathcal{H}^{n-1}(\partial^* E) < \infty$, so E is a set of finite perimeter.

201

Question 2

(a) $F = \frac{x}{|x|^n}$

Let $R > 0$.

$$\int_{B_R(0)} \left|\frac{x}{|x|^n}\right|^p dx = \int_0^R \int_{\partial B_1(0)} \frac{r^{n-1}}{r^{p(n-1)}} \, d\sigma \, dr$$

$$= \mathcal{H}^{n-1}(\partial B_1(0)) \int_0^R r^{n-1-p(n-1)} \, dr$$

$$= \mathcal{H}^{n-1}(\partial B_1(0)) \left. \frac{r^{n-p(n-1)}}{n-p(n-1)} \right]_0^R$$

$$= \mathcal{H}^{n-1}(\partial B_1(0)) \frac{R^{n-p(n-1)}}{n-p(n-1)}$$

want $n - p(n-1) > 0 \iff n - pn + p > 0$

$\iff n - p(n-1) > 0 \iff \boxed{p < \frac{n}{n-1}}$

$\Rightarrow F \in L^p(B_R(0))$ for any $R > 0$

$\Rightarrow F \in L^p_{loc}(\mathbb{R}^N$

We now check that $\operatorname{div} F = n\omega_n \delta_0$

$$\langle \operatorname{div} F, \varphi \rangle = -\int_{\mathbb{R}^n} F \cdot \nabla\varphi = -\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} F \cdot \nabla\varphi \, dx$$

Note:

$$\int_D F \cdot \nabla\varphi = -\int_{\partial D} \varphi F \cdot \nu - \int_D \varphi \operatorname{div} F$$

$$\langle \operatorname{div} F, \varphi \rangle = \lim_{\varepsilon \to 0} \cancel{\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \varphi \operatorname{div} F} + \int_{\partial B_\varepsilon(0)} \varphi F \cdot \nu \, d\mathcal{H}^{n-1}$$

$$= \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(0)} \varphi(x) \frac{x}{|x|^n} \cdot \frac{x}{|x|} \, dx$$

$$= \operatorname{Arctan} \frac{x}{\varepsilon}\Big]_{\varepsilon}^{1-\varepsilon} - \operatorname{Arctan} \frac{x}{1-\varepsilon}\Big]_{\varepsilon}^{1-\varepsilon}$$

$$+ \operatorname{Arctan} \frac{y}{\varepsilon}\Big]_{\varepsilon}^{1-\varepsilon} - \operatorname{Arctan} \frac{y}{1-\varepsilon}\Big]_{\varepsilon}^{1-\varepsilon}$$

$$= \operatorname{Arctan} \frac{1-\varepsilon}{\varepsilon} - \frac{\pi}{4} - \frac{\pi}{4} + \operatorname{Arctan} \frac{\varepsilon}{1-\varepsilon}$$

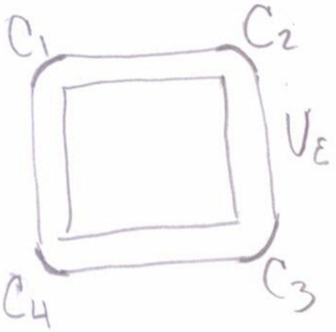
$$+ \operatorname{Arctan} \frac{1-\varepsilon}{\varepsilon} - \frac{\pi}{4} - \frac{\pi}{4} + \operatorname{Arctan} \frac{\varepsilon}{1-\varepsilon}$$

$$= 2 . \operatorname{Arctan} \frac{1-\varepsilon}{\varepsilon} - \pi + 2 \operatorname{Arctan} \frac{\varepsilon}{1-\varepsilon}$$

$$\lim_{\varepsilon \to 0} \int_{U^\varepsilon} F \cdot \nu \, d\mathcal{H}^1 = 2\left(\frac{\pi}{2}\right) - \pi + 2(0) = 0$$

$$\Rightarrow \boxed{0 = \int_U \operatorname{div} F = - \lim_{\varepsilon \to 0} \int_{\partial U^\varepsilon} F \cdot \nu \, d\mathcal{H}^1}$$

interior trace.



$$\partial U_\varepsilon = [(0,1) \times \{-\varepsilon\}] \cup [(0,1) \times \{1+\varepsilon\}]$$

$$\cup [\{-\varepsilon\} \times (0,1)] \cup [\{1+\varepsilon\} \times (0,1)]$$

$$\cup C_1 \cup C_2 \cup C_3 \cup C_4$$

$$\int_{\partial U_\varepsilon} F \cdot \nu \, d\mathcal{H}^1 = \sum_{i=1}^{3} \int_{C_i} F \cdot \nu + \int_{C_4} F \cdot \nu$$

$$+ \int_0^1 \frac{-\varepsilon}{x^2+\varepsilon^2} dx - \int_0^1 \frac{1+\varepsilon}{x^2+(1+\varepsilon)^2} dx + \int_0^1 \frac{-\varepsilon}{y^2+\varepsilon^2} dy$$

$$- \int_0^1 \frac{(1+\varepsilon)\, dy}{y^2+(1+\varepsilon)^2}$$

$$= \sum_{i=1}^{3} \int_{C_i} F\cdot\nu + \int_{C_4} F\cdot\nu - 2\left[\operatorname{Arc\,tan} \frac{1}{\varepsilon}\right] - 2\left[\operatorname{Arc\,tan} \frac{1}{1+\varepsilon}\right]$$

$$= O_\varepsilon(1) + \int_{\pi}^{\frac{3\pi}{2}} (F\cdot\nu)(\vec{r}(\theta))\,\|\vec{r}\,'(\theta)\|\,d\theta \; - \pi - \frac{\pi}{2}$$

$$\vec{r}(\theta) = (\varepsilon\cos\theta,\ \varepsilon\sin\theta)$$

$$= O_\varepsilon(1) + \int_{\pi}^{3/2\,\pi} \left(-\frac{1}{\varepsilon}\right)\cdot\varepsilon\, d\theta - \pi - \frac{\pi}{2}$$

$$= O_\varepsilon(1) - \frac{\pi}{2} - \pi - \frac{\pi}{2}$$

$$\lim_{\varepsilon\to 0} \int_{\partial V_\varepsilon} F\cdot\nu = -2\pi$$

Also: $\int_{\bar{U}} \operatorname{div} F = 2\pi$

$\therefore$ $\boxed{2\pi = \int_{\bar{U}} \operatorname{div} F = -\lim_{\varepsilon\to 0} \int_{\partial V_\varepsilon} F\cdot\nu}$

exterior trace.

Question 3 :

$$\psi(x) = c\left(\frac{r^\gamma}{|x|^\gamma} - 1\right), \quad \text{Barrier}$$

$n = 2$

$$\frac{\partial \psi}{\partial x_1} = -c\gamma r^\gamma \frac{x_1}{|x|^{\gamma+2}}$$

$$\frac{\partial \psi}{\partial x_2} = -c\gamma r^\gamma \frac{x_2}{|x|^{\gamma+2}}$$

$$\frac{\partial^2 \psi}{\partial x_1^2} = -c\gamma r^\gamma \left[\frac{1}{|x|^{\gamma+2}} + x_1\left(\frac{-(\gamma+2)\,x_1}{|x|^{\gamma+4}}\right)\right]$$

$$= -c\gamma r^\gamma \left[\frac{1}{|x|^{\gamma+2}} - (\gamma+2)\frac{x_1^2}{|x|^{\gamma+4}}\right]$$

$$= \frac{-c\gamma r^\gamma}{|x|^{\gamma+2}}\left[1 - (\gamma+2)\frac{x_1^2}{|x|^2}\right]$$

$$\frac{\partial^2 \psi}{\partial x_2^2} = \frac{-c\gamma r^\gamma}{|x|^{\gamma+2}}\left[1 - (\gamma+2)\frac{x_2^2}{|x|^2}\right]$$

$$\frac{\partial^2 \psi}{\partial x_1 \partial x_2} = -c\gamma r^\gamma x_1\left[-(\gamma+2)\frac{x_2}{|x|^{\gamma+4}}\right]$$

$$= \frac{-c\gamma r^\gamma}{|x|^{\gamma+2}}\left[0 - (\gamma+2)\frac{x_1 x_2}{|x|^2}\right]$$