Name: ______________________________

1. Let $E = \cup_{i=1}^{\infty} B(r_i, \frac{1}{2^i})$, where $r_1, r_2, ...$ are all the rational numbers. Show that E is a set of finite perimeter in $\mathbb{R}^n$.
   Hint: If E_k is a sequence of sets of finite perimeter in $\mathbb{R}^n$ and
   $$\chi_{E_k} \to \chi_E \text{ in } L^1(\mathbb{R}^n) \tag{1}$$
   then
   $$P(E) \leq \liminf_{k \to \infty} P(E_k) \tag{2}$$
   Recall our notation $P(E) = \mathcal{H}^{n-1}(\partial^* E)$ and $P(E_k) = \mathcal{H}^{n-1}(\partial^* E_k)$

2. A vector field $F \in L^p(\Omega, \mathbb{R}^n)$, for some $1 \le p \le \infty$, is called a divergence-measure field, and we write $F \in \mathcal{DM}^p(\Omega)$, if the distributional divergence div$F$ is a real finite Radon measure in Ω. A vector field is a locally divergence-measure field, and we write $F \in \mathcal{DM}^p_{loc}(\Omega)$, if the restriction of F to W is in $\mathcal{DM}^p(W)$ for any $W \subset\subset \Omega$.

Consider the vector field $F(x) = \frac{x}{|x|^n}$, $x \in \mathbb{R}^n \setminus 0$.
(a) Show that $F \in \mathcal{DM}^p_{loc}(\mathbb{R}^n)$, for $1 \le p < \frac{n}{n-1}$, and div$F = n\omega_n \delta_0$, where $\omega_n = |B(0,1)|$.
(b) Consider the case $n = 2$; that is, $F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. Hence div$F = 2\pi\delta_0$. Verify with this example that the normal trace distribution for divergence-measure fields in L^p ($p \neq \infty$) can be represented as the limit of classical normal traces over approximations of the domain. More precisely, let $U = (0,1) \times (0,1)$, and d be the signed distance from ∂U and

$$U^\epsilon = \{x \in \mathbb{R}^2 : d(x) > \epsilon\} \quad U_\epsilon = \{x \in \mathbb{R}^2 : d(x) > -\epsilon\}$$

Show that

$$0 = \text{div } F(U) = -\lim_{\epsilon \to 0} \int_{\partial U^\epsilon} F \cdot \nu_{U^\epsilon} \, d\mathcal{H}^1$$

and

$$2\pi = \text{div } F(\overline{U}) = -\lim_{\epsilon \to 0} \int_{\partial U_\epsilon} F \cdot \nu_{U_\epsilon} \, d\mathcal{H}^1$$

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3. The main barrier used to study our free boundary problem is given by

$$\psi(x) = c\left(\frac{r^\gamma}{|x|^\gamma} - 1\right), x \neq 0, \quad r, \gamma, c > 0$$

Show, for $n = 2$, that
(a) If $\gamma \geq \frac{\Lambda - \lambda}{\lambda}$ then $\mathcal{M}^-(\psi) \geq 0$.
(b) If $\gamma \leq \frac{\lambda - \Lambda}{\Lambda}$ then $\mathcal{M}^+(\psi) \leq 0$.
That is, in case (a) ψ is a subsolution of the extremal Pucci minus operator and in case (b) ψ is a supersolution of the extremal Pucci plus operator. The constants λ and Λ are the ellipticity constants of the operators.
Hint: Compute the matrix $D^2\psi(x)$ and write it in the form $I + A$ where I is the identity matrix. Then, the eigenvalues of $I + A$ are $\lambda_1 + 1$ and $\lambda_2 + 1$ where λ_1 and λ_2 are the eigenvalues of A.

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