# FUNCTIONS OF BOUNDED VARIATION AND SETS OF FINITE PERIMETER

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Abstract. We present in these notes some fine properties of functions of bounded variation and sets of finite perimeter, which will be used in the first part of the mini-course.

## 1. Radon and Hausdorff measures

We define

$$\mathcal{M}(\Omega) = \{\mu : \mu \text{ is a Radon measure in } \Omega; \text{i.e.}, \|\mu\|(K) < \infty, \text{ for any } K \subset \Omega \text{ compact set } \}.$$

We denote the total variation of $\mu$ as $\|\mu\|$, which is denoted by

$$\|\mu\|(B) = \sup\{\sum_{i=1}^{\infty} |\mu(C_i)| : B = \cup C_i, C_i \text{ pairwise disjoint}\}$$

We recall that if $\mu$ is a signed measure then, the positive and negative parts of $\mu$ , denoted as $\mu^+$ and $\mu^-$ respectively, are given by

$$\mu^+ = \frac{\|\mu\| + \mu}{2}, \ \mu^- = \frac{\|\mu\| - \mu}{2}.$$

Clearly, we have that

$$\mu \leq \|\mu\|$$

We define now

## 2. Week convergence of measures

**Definition 2.1.** Let $\mu_k, \mu \in \mathcal{M}$. We say that $\mu_k \to \mu$ weakly* if

$$\int_\Omega f d\mu_k \to \int_\Omega f d\mu, \text{ for all } f \in C_c(\Omega).$$

The following result will be of great use to us

**Theorem 2.2.** *Let $\mu_k \in \mathcal{M}(\Omega)$ such that $\sup_k \|\mu_k\|(K) < \infty$, for each compact set $K \in \Omega$. Then, there exists a sequence $\mu_{k_h}$ (which we denote again as $\mu_k$) such that*

$$\mu_k \to \mu \text{ weakly*}.$$

**Lemma 2.3.** *Let $c \in \mathbb{R}$ and let $f : \Omega \to [c, \infty]$ not identically equal to $\infty$. Define, for $t > 0$*

$$f_t(\mathbf{x}) = \inf\{f(\mathbf{y}) + td(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \Omega\},$$

*then $f_t$ is continuous, $f_t \leq f$ and $f_t(\mathbf{x}) \uparrow f(\mathbf{x})$ as $t \uparrow \infty$ whenever $\mathbf{x}$ is a lower semicontinuity point of $f$.*

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*Proof.* Let $\mathbf{x}$ be a lower semicontinuity point of $f$; i.e.,

$$f(\mathbf{x}) \leq \liminf f(\mathbf{x}_i) \tag{2.1}$$

whenever $\mathbf{x}_i \to \mathbf{x}$. Let $\mathbf{x}_t \in \mathbb{R}^N$ such that:

$$f(\mathbf{x}_t) + td(\mathbf{x}, \mathbf{x}_t) < f_t(\mathbf{x}) + \frac{1}{2^t} \tag{2.2}$$

Clearly, $f_t(\mathbf{x}) \leq f(\mathbf{x})$. We will show that $f_t(\mathbf{x}) \uparrow f(\mathbf{x})$ as $t \to \infty$. If $f_t(\mathbf{x}) \uparrow \infty$ as $t \uparrow \infty$ we are done. We now assume that $f_t(\mathbf{x})$ has a finite limit as $t \uparrow \infty$, say $\alpha$. Thus, (2.2) implies that $|\mathbf{x}_t - \mathbf{x}| \to 0$. Since $f$ is lower semicontinuos at $\mathbf{x}$ we obtain, using (2.2):

$$f(\mathbf{x}) \leq \liminf f(\mathbf{x}_t) \leq \lim f_t(\mathbf{x}) = \alpha. \tag{2.3}$$

Since $f_t(\mathbf{x}) \leq f(\mathbf{x})$, the reverse inequality

$$f(\mathbf{x}) \geq \lim f_t(\mathbf{x})$$

also holds. □

**Theorem 2.4.** *Let $\mu, \mu_k \in \mathcal{M}$ such that $\mu_k \to \mu$ weakly*. Then (a) If $\mu_k$ are positive then for every lower semicontinuous function $f : \Omega \to [0, \infty]$*

$$\int_\Omega f d\mu \leq \liminf_{k\to\infty} \int_\Omega f d\mu_k. \tag{2.4}$$

*In particular, if $f = \chi_A$, $A$ open we have*

$$\mu(A) \leq \liminf_{k\to\infty} \mu_k(A), \tag{2.5}$$

*and for every upper semicontinuous function $g : \Omega \to [0, \infty]$ with compact support we have*

$$\limsup_{k\to\infty} \int_\Omega g d\mu_k \leq \int_\Omega g d\mu. \tag{2.6}$$

*In particular, if $g = \chi_K$, $K$ compact, then*

$$\limsup_{k\to\infty} \mu_k(K) \leq \mu(K). \tag{2.7}$$

*(b) If $\|\mu_k\| \to \lambda$ weakly*, then $\|\mu\| \leq \lambda$. Moreover, if $E$ is a $\mu$-measurable set that satisfies $\lambda(\partial E) = 0$, then*

$$\mu_k(E) \to \mu(E). \tag{2.8}$$

*More generally:*

$$\int_\Omega u d\mu = \lim_{k\to\infty} \int_\Omega u d\mu_k. \tag{2.9}$$

*for any bounded Borel function $u : \Omega \to \mathbb{R}$ with compact support such that the set of its discontinuities is $\mu$- negligible.*

*Proof.* Let $f : \Omega \to [0, \infty]$ be a lower semicontinuous function and let $f_t$ be the approximation given by the previous lemma, such that $f_t(\mathbf{x}) \uparrow f(\mathbf{x})$. Choose a function $\psi$ so that $0 \leq \psi \leq 1$ and $\psi \in C_c(\Omega)$. Since $\mu_k \to \mu$ weakly* we have that

$$\begin{aligned} \int_\Omega \psi f_t d\mu &= \lim_{k\to\infty} \int_\Omega \psi f_t d\mu_k \\ &\leq \liminf_{k\to\infty} \int_\Omega f_t d\mu_k. \end{aligned}$$

We now take the sup over all $\psi$ and let $t \to \infty$ to obtain

$$\int_\Omega f d\mu \leq \liminf_{k\to\infty} \int_\Omega f d\mu_k.$$

The proof of the case when $g$ is upper semicontinuous is similar but using the corresponding approximation as in the previous lemma. □

**Definition 2.5.** If $\mu \in \mathcal{M}(\Omega)$ we define

$$\text{supp } \mu = \{\mathbf{y} \in \Omega : \mu(B(\mathbf{y}, r)) > 0, \text{ for all } r > 0\}$$

**Theorem 2.6.** *Let $\nu$ be a non-negative locally finite Radon measure on $\Omega$. Let $J$ be a set with the following properties*
*(a) For all $\mathbf{y} \in J$ there exist orthonormal coordinates $x_1, x_2, ..., x_N$ such that $C_\mathbf{y} := \{8|x_1| \geq |(x_2, x_3, ..., x_N)|\}$ satisfies*

$$\lim_{r\to 0} \frac{\nu(\mathbf{y} + C_\mathbf{y}) \cap B(\mathbf{y}, r)}{r^{N-1}} = 0, \tag{2.10}$$

*and*
*(b) For all $\mathbf{y} \in J$*

$$\liminf_{r\to 0} \frac{\nu(B(\mathbf{y}, r))}{r^{N-1}} > 0. \tag{2.11}$$

*Then $J$ is contained in a countably union of Lipschitz graphs.*

**Definition 2.7.** Let $\mu \in \mathcal{M}(\mathbb{R}^N)$. The rescalings or blow-ups of the measure $\mu$ around $\mathbf{y}$ is the sequence of measures

$$\mu_{\mathbf{y},r}(A) := \frac{\mu(\mathbf{y} + rA)}{r^{N-1}}, \; A \subset \mathbb{R}^N \tag{2.12}$$

We have

**Lemma 2.8.** *For $\mathcal{H}^{N-1}$ -almost every $\mathbf{y} \in \mathbb{R}^N$, the sequence $\mu_{\mathbf{y},r}$ has a weakly* convergence subsequence. (A limit will be denoted as $\mu_{\mathbf{y},\infty}$).*

*Proof.* We claim that

$$\limsup_{r\to 0} \frac{\mu(B(\mathbf{y}, r))}{r^{N-1}} < \infty, \text{ for } \mathcal{H}^{N-1} - a.e. \; \mathbf{y} \in \mathbb{R}^N. \tag{2.13}$$

The claim implies that, for each compact set $K$,

$$\sup_r \mu_{\mathbf{y},r}(K) < \infty,$$

and thus the Lemma follows. In order to prove the claim we proceed by contradiction and assume that there exists a bounded set $K$ with $\mathcal{H}^{N-1}(K) > 0$ and

$$\limsup_{r\to 0} \frac{\mu(B(\mathbf{y}, r))}{r^{N-1}} = \infty, \text{ for all } \mathbf{y} \in K.$$

□

We choose an open set $K_\varepsilon$, $K \subset K_\varepsilon$ and $\mu(K_\varepsilon) < \mu(K) + \varepsilon$. Since $K_\varepsilon$ is open, for each $\mathbf{y} \in K_\varepsilon$, there exists an open ball $B_{r_\mathbf{y}}(\mathbf{y}) \subset K_\varepsilon$, $r_\mathbf{y} \leq r_0$ and $\mu(B_{r_\mathbf{y}}(\mathbf{y})) \geq M r_\mathbf{y}^{N-1}$. The family of balls $\{B_{r_\mathbf{y}}(\mathbf{y})\}$ covers $K_\varepsilon$. Thus, Vitali's covering Lemma implies that there exists a countable collection $\{B_{r_i}(\mathbf{y}_i)\}$ such that $\cup B_{5r_i}(\mathbf{y}_i)$ covers $K_\varepsilon$. Therefore, we have that $\mu(K_\varepsilon) \geq \sum \mu(B_{r_i}(\mathbf{y}_i)) =$

$M \sum r_i^{N-1} = C(N)\mathcal{H}^{N-1}(K)$, which leads to the contradiction $\mu(K_\varepsilon) = \infty$, since $M$ is arbitrary.

**Lemma 2.9.** *Let $\mathbb{R}^N_+ = \{\mathbf{x} \in \mathbb{R}^N : x_n > 0\}$ and let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^N_+$. Let $\Omega \subset \mathbb{R}^{N-1}$ be a bounded open set. For $r > 0$ and $y \in \Omega$, let*

$$C_r^R(y) = (y + (-rR, rR)^d) \times (0, rR).$$

*then, for $\mathcal{H}^{N-1}$ -almost every $y \in \Omega$,*

$$\lim_{r\to 0} \frac{\mu(C_r^R(y))}{r^{N-1}} = 0$$

*Proof.* Let

$$A_k = \{y \in \Omega : \limsup_{r\to 0} \frac{\mu(C_r^R(y))}{r^{N-1}} > \frac{1}{k}\}.$$

It is sufficient to show that $\mathcal{H}^{N-1}(A_k) = 0$ for each $k$. Given $y \in A_k$ and $\varepsilon > 0$, there exists a number $r_y < \varepsilon$ such that

$$\mu\left(C_{r_y}^R(y)\right) > \frac{1}{2k} r_y^{N-1}.$$

Using a covering argument we can choose a sequence $y_i \in \Omega$ such that $\mathcal{B}(y_i, r_i) := \{y \in \Omega : \|y - y_i\| < r_i\}$ where $r_i = r_{y_i}$ are disjoints and $A_k \subset \cup_{i=1}^\infty \mathcal{B}(y_i, 3r_i)$. Then

$$\mathcal{H}^{N-1}(A_k) \leq \omega_{N-1} \sum_{i=1}^\infty (3r_i)^{N-1} \leq 2k\omega_{N-1} 3^{N-1} \sum_{i=1}^\infty \mu\left(C_{r_i}^R(y_i)\right).$$

On the other hand, since $r_i < \varepsilon$, we have

$$C_{r_i}^R(y_i) \subset \Omega_\varepsilon = \Omega \times (0, \varepsilon R)$$

and hence

$$\mathcal{H}^{N-1}(A_k) < 2k\omega_{N-1} 3^{N-1} \mu(\Omega_\varepsilon) \text{ for all } \varepsilon > 0.$$

Since $\mu$ is a Radon measure we have $\mu(\Omega_\varepsilon) \to 0$ as $\varepsilon \to 0$ and $\mathcal{H}^{N-1}(A_k) = 0$ □

## 3. Convolutions and representation of $BV$ functions

**Definition 3.1.** Let $E, F$ sets in $\Omega$ and let $\mathbf{x} \in \Omega$. We say that $\mathbf{x}$ is a point of density $\alpha$ for $E$ relative to $F$ if

$$\lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap E \cap F|}{|B_r(\mathbf{x}) \cap F|} = \alpha.$$

**Definition 3.2.** Let $f$ be a measurable function. We say that $\beta$ is the approximate limit of $f(\mathbf{x})$ as $\mathbf{x} \to \mathbf{x}_0$ if $\forall \varepsilon > 0$, $\mathbf{x}_0$ is a point of density 1 for the set $\{\mathbf{x} : |f(\mathbf{x}) - \beta| < \varepsilon\}$. It is denoted by $\operatorname{alim}_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \beta$.

The above definition says that if $\operatorname{alim}_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \beta$ then $\forall \varepsilon > 0$:

$$\lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap \{|f(\mathbf{x}) - \beta| < \varepsilon\}|}{|B_r(\mathbf{x})|} = 1.$$

**Definition 3.3.** We say that $\operatorname{alim}_{\mathbf{x}\to\mathbf{x}_0, \mathbf{x}\in E} f(\mathbf{x}) = \beta$ if $\forall \varepsilon > 0$, $\mathbf{x}_0$ is a point of density 1 for the set $\{|f(\mathbf{x}) - \beta| < \varepsilon\}$, relative to $E$; that is, if $\forall \varepsilon > 0$

$$\lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap \{|f(\mathbf{x}) - \beta| < \varepsilon\} \cap E|}{|B_r(\mathbf{x}) \cap E|} = 1.$$

Let $w \in \Omega$. We introduce the following notation:

$$\pi_w(\mathbf{x}_0) = \{(\mathbf{x} - \mathbf{x}_0, w) > 0\}$$

and

$$f_w(\mathbf{x}_0) = \operatorname*{alim}_{\mathbf{x}\to\mathbf{x}_0, \mathbf{x}\in\pi_w(\mathbf{x}_0)} f(\mathbf{x}).$$

**Definition 3.4.** $\mathbf{x}_0$ is a regular point for $f(\mathbf{x})$ if there exists $w \in \mathbb{R}^N$ such that $f_w(\mathbf{x}_0)$ and $f_{-w}(\mathbf{x}_0)$ exist. The vector $w$ is called a defining vector for $f$ at $\mathbf{x}_0$.

It is easy to check that if $w$ is a defining vector for $f$ at $\mathbf{x}_0$ and $f_w(\mathbf{x}_0) = f_{-w}(\mathbf{x}_0)$, then any $\tilde{w} \in \mathbb{R}^N$ is also a defining vector and

$$f_w(\mathbf{x}_0) = f_{\tilde{w}}(\mathbf{x}_0) = f_{-\tilde{w}}(\mathbf{x}_0).$$

On the other hand, if $w$ is a defining vector for $f$ at $\mathbf{x}_0$ and $f_w(\mathbf{x}_0) \neq f_{-w}(\mathbf{x}_0)$ then $w$ is uniquely determined up to sign.

We have the following.

**Theorem 3.5.** *Let $f \in BV(\mathbb{R}^N)$. Then, $\mathcal{H}^{N-1} - a.e.$ $\mathbf{x} \in \mathbb{R}^N$ is a regular point for $f$.*

Let $\rho$ be a standard symmetric mollifier and $\rho_r = \dfrac{1}{r^N}\,\rho\left(\dfrac{\mathbf{x}}{r}\right)$.

**Definition 3.6.**

$$\bar{f}_r(\mathbf{x}) = \int_{\mathbb{R}^N} \rho_r(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}$$

and

$$\bar{f}(\mathbf{x}) = \lim_{r\to 0}\ \bar{f}_r(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^N,$$

whenever the above limit exists. We have:

**Lemma 3.7.** *If $\mathbf{x}$ is a regular point for $f$ and $w$ is a defining vector then:*

$$\bar{f}(\mathbf{x}) = \frac{1}{2}(f_w(\mathbf{x}) + f_{-w}(\mathbf{x})).$$

*Proof.*

$$\begin{aligned}\bar{f}_r(\mathbf{x}) &= \int_{\mathbb{R}^N} \rho_r(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \\ &= \int_{\mathbb{R}^N\cap\pi_w} \rho_r(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \\ &\qquad + \int_{\mathbb{R}^N\cap\pi_{-w}} \rho_r(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}.\end{aligned}$$

Let us compute first

$$\int_{\mathbb{R}^N\cap\pi_w} \rho_r(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}.$$

We have:

$$\begin{aligned}\int_{\mathbb{R}^N\cap\pi_w} \rho_r(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} &= \int_{\mathbb{R}^N\cap\pi_w} \rho_r(\mathbf{x} - \mathbf{y}) \left(f(\mathbf{y}) - f_w(\mathbf{x})\right)\, d\mathbf{y} \\ &\quad + \int_{\mathbb{R}^N\cap\pi_w} \rho_r(\mathbf{x} - \mathbf{y}) f_w(\mathbf{x})\, d\mathbf{y}\end{aligned}$$

$$\int_{\mathbb{R}^N\cap\pi_w} \rho_r(\mathbf{x}-\mathbf{y})f_w(\mathbf{x})\,d\mathbf{y} = f_w(\mathbf{x})\int_{B_r(\mathbf{x})\cap\pi_w}\rho_r(\mathbf{y})\,d\mathbf{y}$$
$$= \frac{f_w(\mathbf{x})}{2};$$

because $\int_{B_r(\mathbf{x})}\rho_r(\mathbf{y})\,d\mathbf{y} = 1.$

Also, if we define $A^\varepsilon = \{\mathbf{y} : |f(\mathbf{y}) - f_w(\mathbf{x})| < \varepsilon\}$ we obtain

$$\int_{\mathbb{R}^N\cap\pi_w} \rho_r(\mathbf{x}-\mathbf{y})\left(f(\mathbf{y})-f_w(\mathbf{x})\right)\,d\mathbf{y} = \int_{B_r(\mathbf{x})\cap\pi_w}\rho_r(\mathbf{y})\left(f(\mathbf{y})-f_w(\mathbf{x})\right)\,d\mathbf{y}$$
$$= \int_{B_r(\mathbf{x})\cap\pi_w\cap A^\varepsilon}\rho_r(\mathbf{y})\left(f(\mathbf{y})-f_w(\mathbf{x})\right)\,d\mathbf{y} + \int_{B_r(\mathbf{x})\cap\pi_w\cap(A^\varepsilon)^c}\rho_r(\mathbf{y})\left(f(\mathbf{y})-f_w(\mathbf{x})\right)\,d\mathbf{y}$$
$$\leq \frac{C}{r^N}\,\varepsilon\,|B_r(\mathbf{x})\cap\pi_w\cap A^\varepsilon| + \frac{2M}{r^N}|B_r(\mathbf{x})\cap\pi_w\cap(A^\varepsilon)^c|.$$

Since:

$$\lim_{r\to 0}\frac{|B_r(\mathbf{x})\cap A^\varepsilon\cap\pi_w|}{|B_r(\mathbf{x})\cap\pi_w|} = 1$$

and

$$\lim_{r\to 0}\frac{|B_r(\mathbf{x})\cap\pi_w\cap(A^\varepsilon)^c|}{|B_r(\mathbf{x})\cap\pi_w|} = 0$$

we conclude, since $\varepsilon$ is arbitrary, that:

$$\lim_{r\to 0}\int_{\mathbb{R}^N\cap\pi_w}\rho_r(\mathbf{x}-\mathbf{y})(f(\mathbf{y})-f_w(\mathbf{x}))\,d\mathbf{y} = 0$$

and thus:

$$\lim_{r\to 0}\int_{\mathbb{R}^N\cap\pi_w}\rho_r(\mathbf{x}-\mathbf{y})f(\mathbf{y})\,d\mathbf{y} = \frac{f_w(\mathbf{x})}{2}.$$

In the same way we can show that:

$$\lim_{r\to 0}\int_{\mathbb{R}^N\cap\pi_{-w}}\rho_r(\mathbf{x}-\mathbf{y})f(\mathbf{y})\,d\mathbf{y} = \frac{f_{-w}(\mathbf{x})}{2}.$$

We conclude:

$$\bar{f}(\mathbf{x}) = \frac{1}{2}(f_w(\mathbf{x}) + f_{-w}(\mathbf{x})).$$

□

## 4. Sets of Finite Perimeter

## 5. Basic properties of sets of finite perimeter

In what follows we will work in $\mathbb{R}^N$. We introduce now a few basic definitions and results on the theory of functions of bounded variations and sets of finite perimeter, for which we refer mainly to [1], [12] and [15] (see also [14] and [21]).

**Definition 5.1.** A function $u \in L^1(\mathbb{R}^N)$ is called a *function of bounded variation* if $Du$ is a finite $\mathbb{R}^N$-vector valued Radon measure on $\mathbb{R}^N$. A measurable set $E \subset \mathbb{R}^N$ is called a *set of finite perimeter* in $\mathbb{R}^N$ (or a Caccioppoli set) if $\chi_E \in BV(\mathbb{R}^N)$. Consequently, $D\chi_E$ is an $\mathbb{R}^N$-vector valued Radon measure on $\mathbb{R}^N$ whose total variation is denoted as $\|D\chi_E\|$.

By the polar decomposition of measures, we can write $D\chi_E = \nu_E \|D\chi_E\|$, where $\nu_E$ is a $\|D\chi_E\|$-measurable function such that $|\nu_E(x)| = 1$ for $\|D\chi_E\|$-a.e. $x \in \mathbb{R}^N$.

We define the perimeter of $E$ as

$$P(E) := \sup \left\{ \int_E \operatorname{div}(\varphi)\, dx : \varphi \in C_c^1(\mathbb{R}^N; \mathbb{R}^N), \|\varphi\|_\infty \leq 1 \right\}$$

and it can be proved that $P(E) = \|D\chi_E\|(\mathbb{R}^N)$.

The notion of perimeter generalizes the idea of $\mathcal{H}^{N-1}$-measure of the boundary of the set $E$. It is a well-known fact that the topological boundary of a set of finite perimeter can be very irregular, it can even have full Lebesgue measure. This suggests that for a set of finite perimeter is interesting to consider subsets of $\partial E$ instead. De Giorgi considered a set of finite $\mathcal{H}^{N-1}$-measure on which $\|D\chi_E\|$ is concentrated, which he called reduced boundary.

**Definition 5.2.** We say that $x \in \partial^* E$, the *reduced boundary* of $E$, if

(1) $\|D\chi_E\|(B(x,r)) > 0, \forall r > 0$;
(2) $\lim_{r \to 0} \frac{1}{\|D\chi_E\|(B(x,r))} \int_{B(x,r)} \nu_E(y)\, d\|D\chi_E\|(y) = \nu_E(x)$;
(3) $|\nu_E(x)| = 1$.

It can be shown that this definition implies a geometrical characterization of the reduced boundary, by using the blow-up of the set $E$ around a point of $\partial^* E$.

**Theorem 5.3.** *If $x \in \partial^* E$, then*

$$\frac{E - x}{\varepsilon} \to H^+_{\nu_E}(x) := \{y \in \mathbb{R}^N : y \cdot \nu_E(x) \geq 0\} \quad in \quad L^1_{\mathrm{loc}}(\mathbb{R}^N) \quad as \quad \varepsilon \to 0$$

*and*

$$\frac{(\mathbb{R}^N \setminus E) - x}{\varepsilon} \to H^-_{\nu_E}(x) := \{y \in \mathbb{R}^N : y \cdot \nu_E(x) \leq 0\} \quad in \quad L^1_{\mathrm{loc}}(\mathbb{R}^N) \quad as \quad \varepsilon \to 0.$$

This theorem will be proved later. Formulated in another way, for $\varepsilon > 0$ small enough, $E \cap B(x, \varepsilon)$ is asymptotically close to the half ball $H^-_{\nu_E}(x) \cap B(x, \varepsilon)$.

Because of this result, we call $\nu_E(x)$ *measure theoretic unit interior normal* to $E$ at $x \in \partial^* E$, since it is a generalization of the concept of unit interior normal.

In addition, De Giorgi proved that $\|D\chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^* E$, so that $D\chi_E = \nu_E \mathcal{H}^{N-1} \llcorner \partial^* E$ and $P(E) = \mathcal{H}^{N-1}(\partial^* E)$ (see [12, Section 5.7.3, Theorem 2]).

For every $\alpha \in [0, 1]$ we set

$$E^\alpha := \{x \in \mathbb{R}^N : D(E, x) = \alpha\},$$

where

$$D(E, x) := \lim_{r \to 0} \frac{|B(x,r) \cap E|}{|B(x,r)|},$$

and we give the following definitions:

(1) $E^1$ is called the *measure theoretic interior* of $E$.
(2) $E^0$ is called the *measure theoretic exterior* of $E$.

We recall (see Maggi [15, Example 5.17]) that every Lebesgue measurable set is equivalent to the set of its points of density one; that is,

$$\mathcal{L}^N(E \Delta E^1) = \mathcal{L}^N((\mathbb{R}^N \setminus E) \Delta E^0) = 0. \tag{5.1}$$

It is also a well-know result due to Federer that there exists a set $\mathcal{N}$ with $\mathcal{H}^{N-1}(\mathcal{N}) = 0$ such that $\mathbb{R}^N = E^1 \cup \partial^* E \cup E^0 \cup \mathcal{N}$ (see [1, Theorem 3.61]).

The perimeter $P(E)$ of $E$ is invariant under modifications by a set of $\mathcal{L}^N$-measure zero, even though these modifications might largely increase the size of the topological boundary. In this paper we consider the following representative

$$E := E^1 \cup \partial^* E. \tag{5.2}$$

Given a smooth nonnegative radially symmetric mollifier $\rho \in C_c^\infty(B_1(0))$, we denote the mollification of $\chi_E$ by $u_k(x) := (\chi_E * \rho_{\varepsilon_k})(x)$ for some positive sequence $\varepsilon_k \to 0$. We define, for $t \in (0,1)$,

$$A_{k;t} := \{u_k > t\}. \tag{5.3}$$

By Sard's theorem (for which we refer to [15, Lemma 13.15]), we know that, since $u_k : \mathbb{R}^N \to \mathbb{R}$ is $C^\infty$, $\mathcal{L}^1$-a.e. $t \in (0,1)$ is not the image of a critical point for $u_k$ and so $A_{k;t}$ has a smooth boundary for these values of $t$. Thus, for each $k$ there exists a set $Z_k \subset (0,1)$, with $\mathcal{L}^1(Z_k) = 0$, which is the set of values of $t$ for which $A_{k;t}$ has not a smooth boundary. If we set $Z := \bigcup_{k=1}^{+\infty} Z_k$, then $\mathcal{L}^1(Z) = 0$ and, for each $t \in (0,1) \setminus Z$ and for each $k$, $A_{k;t}$ has a smooth boundary.

It is a well-known result from BV theory (see for instance [1, Corollary 3.80]) that every function of bounded variations $u$ admits a representative which is the pointwise limit $\mathcal{H}^{N-1}$-a.e. of any mollification of $u$ and which coincides $\mathcal{H}^{N-1}$-a.e. with the precise representative $u^*$:

$$u^*(x) := \begin{cases} \lim\limits_{r\to 0} \dfrac{1}{|B(x,r)|} \displaystyle\int_{B(x,r)} u(y)\, dy & \text{if this limit exists} \\ 0 \quad \text{otherwise} \end{cases}.$$

For any set of finite perimeter $E$, we denote the precise representative of the function $\chi_E$ by $u_E$, which is given by

$$u_E(x) = \begin{cases} 1, & x \in E^1 \\ 0, & x \in E^0 \\ \frac{1}{2}, & x \in \partial^* E \end{cases}.$$

Since $\mathcal{H}^{N-1}(\mathbb{R}^N \setminus (E^1 \cup \partial^* E \cup E^0)) = 0$, the function $u_E$ is well defined $\mathcal{H}^{N-1}$-a.e..

In order to prove Theorem 9.1, we need to use the classical coarea formula, for which we refer to [12, Section 3.4, Theorem 1].

**Theorem 5.4.** *Let $u : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz. Then, for any $\mathcal{L}^N$-measurable set $A$, we have*

$$\int_A |\nabla u|\, dx = \int_{\mathbb{R}} \mathcal{H}^{N-1}(A \cap u^{-1}(t))\, dt. \tag{5.4}$$

## 6. Structure of sets of finite perimeter

**Definition 6.1.** Let $E$ be a set of finite perimeter and $\mathbf{x} \in \partial^m E$. We say that $\nu$ is the inner unit normal at $\mathbf{x}$ if:

$$\lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap E \cap \pi_\nu|}{|B_r(\mathbf{x}) \cap \pi_\nu|} = 1 \tag{6.1}$$

and

$$\lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap E \cap \pi_{-\nu}|}{|B_r(\mathbf{x}) \cap \pi_{-\nu}|} = 0 \tag{6.2}$$

We know that if $\mathbf{x} \in \partial^* E$, then

$$\nu(\mathbf{x}) = \lim_{r\to 0} \frac{D\chi_E(B_r(\mathbf{x}))}{\|D\chi_E\|(B_r(\mathbf{x}))} \tag{6.3}$$

exists and $\|\nu(\mathbf{x})\| = 1$. We now proceed to show that $\nu(\mathbf{x})$ is the inner normal to $E$ at $\mathbf{x}$. We need first the following result:

**Lemma 6.2.** *Let $E$ be a set of finite perimeter and $\mathbf{x} \in \partial^* E$. Then, there exists $r_0$, that depends on $\mathbf{x}$, such that for all $r \le r_0$;*

*(a)* $\|D\chi_E\|(B_r(\mathbf{x})) \le C_1 r^{N-1}$

*(b)* $|E \cap B_r(\mathbf{x})| \ge C_2 r^N$

*(c)* $|(\mathbb{R}^N \setminus E) \cap B_r(\mathbf{x})| \ge C_3 r^N$,

*where $C_1, C_2, C_3$ are universal constants.*

*Proof.* We recall that if $f \in BV(\Omega)$, where $\Omega$ is an open set with Lipschitz boundary, then:

$$\int_\Omega f \operatorname{div}\varphi = -\int_\Omega \varphi \cdot Df + \int_{\partial\Omega} f_{tr} < \varphi, \mathbf{n} > d\mathcal{H}^{N-1}, \tag{6.4}$$

for all $\varphi \in C_0^1(\mathbb{R}^N)$. Here, $f_{tr}$ is the trace of $f$ on $\partial\Omega$.

If we apply (6.4) to $f = \chi_E$, $\Omega = B(\mathbf{x}, r) = B_r$ and $\varphi \equiv 1$ we have, for a.e. $r$,

$$0 = -\int_{B_r} D\chi_E + \int_{\partial B_r} \chi_E < 1, \mathbf{n} > d\mathcal{H}^{N-1} \tag{6.5}$$

and thus:

$$\left\| \int_{B_r} D\chi_E \right\| \le \mathcal{H}^{N-1}(E \cap \partial B_r) \tag{6.6}$$

On the other hand, since $\mathbf{x} \in \partial^* E$ we have

$$\lim_{r\to 0} \left\| \frac{\int_{B_r} D\chi_E}{\int_{B_r} \|D\chi_E\|} \right\| = \|\nu(\mathbf{x})\| = 1 \tag{6.7}$$

which implies that for r small enough:

$$\left\| \int_{B_r} D\chi_E \right\| \ge \frac{1}{2} \int_{B_r} \|D\chi_E\| \tag{6.8}$$

From (6.8) and (6.6) we have:

$$\|D\chi_E\|(B_r) \le 2\mathcal{H}^{N-1}(E \cap \partial B_r) \le C_1 r^{N-1} \tag{6.9}$$

which gives (a).
The isoperimetric inequality implies:

$$\begin{aligned} |E \cap B_r| &\le C\left(D\chi_{E\cap B_r}(\mathbb{R}^N)\right)^{\frac{N}{N-1}} \\ &= C\left(\|D\chi_E(B_r)\| + \mathcal{H}^{N-1}(E \cap \partial B_r)\right)^{\frac{N}{N-1}} \\ &\le C(\mathcal{H}^{N-1}(E \cap \partial B_r))^{\frac{N}{N-1}} \end{aligned} \tag{6.10}$$

by (6.9)
We define

$$V(r) = |E \cap B_r| \tag{6.11}$$

We have

$$V(r) = \int_{B_r} \chi_E = \int_0^r \int_{\partial B_t} \chi_E d\mathcal{H}^{N-1} dt \tag{6.12}$$

and thus

$$V'(r) = \int_{\partial B_r} \chi_E d\mathcal{H}^{N-1} = \mathcal{H}^{N-1}(E \cap \partial B_r) \tag{6.13}$$

From (6.10) and (6.13) we obtain:

$$\begin{aligned} V(r) &\leq C(V'(r))^{\frac{N}{N-1}} \\ C &\leq V'(r)(V(r))^{-\frac{N-1}{N}} \\ C &\leq (V(r)^{\frac{1}{N}})' \\ V(r) &\geq C_2 r^N, \end{aligned}$$

Which gives (b)

We can obtain (c) in a similar way but using $X_{\mathbb{R}^N \setminus E}$ instead of $X_E$. □

We have the following

**Lemma 6.3.** *Let $E$ be a set of finite perimeter and $\mathbf{x} \in \partial^* E$. Then, $\nu(\mathbf{x})$is the inner unit normal.*

*Proof.* We can assume without loss of generality that $\mathbf{x} = 0$. For $0 \leq r \leq 1$ we define the sets

$$E_r = \frac{E}{r}. \tag{6.14}$$

We fix $R > 0$. Using (a) from lemma 6.2, we have

$$\|D\chi_{E_r}\|(B_r) = r^{1-N} \|D\chi_E\|(B_{rR}) \leq Cr^{1-N}(rR)^{1-N} = CR^{1-N} \tag{6.15}$$

Since 6.15 is true for any $R > 0$, it follows that there exists a set Q such that

$$E_r \to Q \text{ in } L^1_{loc}.$$

We now proceed to show that $\partial Q$ is a hyperplane orthogonal to $\nu$.

We now claim that $Q = \pi_\nu$. We proceed by contradiction and assume first that there exists $\alpha > 0$ such that

$$Q \cap B_\alpha = \emptyset. \tag{6.16}$$

In this case we have, using (b) from lemma 6.2 ,

$$|E_r \cap B_\alpha| = r^{-N}|E \cap B_{r\alpha}| \geq Cr^{-N}(r\alpha)^N = C\alpha^N \tag{6.17}$$

Since $\|E_r \cap B_\alpha\| \to \|Q \cap B_\alpha\|$ and $\|E_r \cap B_\alpha\| \geq C$ it follows that $\|Q \cap B_\alpha\| \geq C$, which contradicts (6.16). If we now assume that there exists $\alpha > 0$ now such that

$$(\mathbb{R}^N \setminus Q) \cap B_\alpha = \emptyset, \tag{6.18}$$

procceeding in the same way but using (c) from lemma 6.2 we obtain a contradiction. This shows that $Q = \pi_\nu$. We can now show that $\nu$ is the inner unit normal to $E$ at $\mathbf{x}$. In fact we have, since $Q = \pi_\nu$,

$$|E_r \cap B_1 \cap \pi_\nu| \to |B_1 \cap \pi_\nu| \tag{6.19}$$

On the other hand

$$|E_r \cap B_1 \cap \pi_\nu| = \frac{|E \cap B_r \cap \pi_\nu|}{r^N}. \tag{6.20}$$

Thus, from (6.19) and (6.20) we have that:

$$\lim_{r\to 0} \frac{|E \cap B_r \cap \pi_\nu|}{r^N} = |B_1 \cap \pi_\nu|, \tag{6.21}$$

or equivalently,

$$\lim_{r\to 0} \frac{|E \cap B_r \cap \pi_\nu|}{|B_r \cap \pi_\nu|} = 1. \tag{6.22}$$

In the same way we show that

$$\lim_{r\to 0} \frac{|E \cap B_r \cap \pi_-\nu|}{|B_r \cap \pi_-\nu|} = 0. \tag{6.23}$$

□

If $E$ is a set of finite perimeter, we know that $\chi_E \in BV(\mathbb{R}^N)$ and thus, from Theorem 3.5 it follows that $\mathcal{H}^{N-1} - a.e.\mathbf{x} \in \mathbb{R}^N$ is a regular point for $\chi_E$. If $u_k$ is a sequence of mollifiers for $\chi_E$, Lemma 3.7 says that if $\mathbf{x} \in \mathbb{R}^N$ is a regular point for $\chi_E$ with defining vector $w$ then the following limit exists

$$\overline{\chi}_E(\mathbf{x}) = \lim_{k\to\infty} u_k(\mathbf{x}) = \frac{1}{2}((\chi_E)_w(\mathbf{x}) + (\chi_E)_{-w}(\mathbf{x})). \tag{6.24}$$

From now on we define

$$u_E(\mathbf{x}) := \overline{\chi}_E(\mathbf{x}). \tag{6.25}$$

If $\mathbf{x} \in E^0$ we have, for any $\varepsilon > 0$,

$$\lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap \{|\chi_E - 0| < \varepsilon\}|}{|B_r(\mathbf{x})|} = \lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap E^c|}{|B_r(\mathbf{x})|} = 1, \tag{6.26}$$

which by definition implies that $a\lim_{\mathbf{x}\to\mathbf{x}} \chi_E = 0$ and thus $\overline{u}_E(\mathbf{x}) = 0$.

If $\mathbf{x} \in E^1$ we have, for any $\varepsilon > 0$,

$$\lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap \{|\chi_E - 1| < \varepsilon\}|}{|B_r(\mathbf{x})|} = \lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap E|}{|B_r(\mathbf{x})|} = 1, \tag{6.27}$$

which by definition implies that $a\lim_{\mathbf{x}\to\mathbf{x}} \chi_E = 1$ and hence $\overline{u}_E(\mathbf{x}) = 1$.

If $\mathbf{x} \in \partial^* E$ we recall that $\nu_E(\mathbf{x}) = \nu(\mathbf{x}) = \nu$ is the inner unit normal to $E$ at

$\mathbf{x}$.
Since, by Lemma 6.3,

$$\lim_{r\to 0}\frac{|B_r(\mathbf{x})\cap\{|\chi_E-1|<\varepsilon\}\cap\pi_\nu|}{|B_r(\mathbf{x})\cap\pi_\nu|}=1 \tag{6.28}$$

and

$$\lim_{r\to 0}\frac{|B_r(\mathbf{x})\cap\{|\chi_E-0|<\varepsilon\}\cap\pi_{-\nu}|}{|B_r(\mathbf{x})\cap\pi_{-\nu}|}=1 \tag{6.29}$$

it follows that $a\lim_{\mathbf{x}\to\mathbf{x},\mathbf{x}\in\pi_\nu}\chi_E=(\chi_E)_\nu(\mathbf{x})$ and $a\lim_{\mathbf{x}\to\mathbf{x},\mathbf{x}\in\pi_{-\nu}}\chi_E=(\chi_E)_{-\nu}(\mathbf{x})=0$. Thus, we obtain

$$u_E(\mathbf{x})=\frac{1}{2}((\chi_E)_\nu(\mathbf{x})+(\chi_E)_{-\nu}(\mathbf{x}))=\frac{1}{2}(1+0)=\frac{1}{2}. \tag{6.30}$$

Since $\mathcal{H}^{N-1}(\partial^m E\setminus\partial^* E)=0$ and

$$\mathbb{R}^N=E^0\cup E^1\cup\partial^* E\cup(\partial^m E\setminus\partial^* E), \tag{6.31}$$

We can finally give the following result

**Lemma 6.4.** *The function $u_E$ is defined $\mathcal{H}^{N-1}$ almost everywhere in $\mathbb{R}^N$*

We have the following result:

**Lemma 6.5.** *Let $E$ be a set of finite perimeter.*
*Then:*

$$\mathcal{H}^{N-1}(\partial^m E\setminus\partial^* E)=0 \tag{6.32}$$

*Proof.* We first note that if $\mathbf{x}\in\mathbb{R}^N$ satisfies

$$\lim_{r\to 0}\frac{\|D\chi_E\|(B_r(\mathbf{x}))}{r^{N-1}}=0 \tag{6.33}$$

then $\mathbf{x}\in E^0$ or $\mathbf{x}\in E^1$. In fact, from the isoperimetric inequality

$$(|D\chi_E|(B_r(\mathbf{x})))^{\frac{N}{N-1}}\geq C\min\{|E\cap B_r(\mathbf{x})|,|(\mathbb{R}^N\setminus E)\cap B_r(\mathbf{x})|\} \tag{6.34}$$

it follows that

$$\left(\frac{\|D\chi_E\|(B_r(\mathbf{x}))}{r^{N-1}}\right)^{\frac{N}{N-1}}\geq C\frac{|E\cap B_r(\mathbf{x})|}{r^N} \tag{6.35}$$

or

$$\left(\frac{\|D\chi_E\|(B_r(\mathbf{x}))}{r^{N-1}}\right)^{\frac{N}{N-1}}\geq C\frac{|(\mathbb{R}^N\setminus E)\cap B_r(\mathbf{x})|}{r^N}; \tag{6.36}$$

and from (6.33) we have

$$\lim_{r\to 0}\frac{|E\cap B_r(\mathbf{x})|}{r^N}=0 \tag{6.37}$$

or

$$\lim_{r\to 0}\frac{|(\mathbb{R}^N\setminus E)\cap B_r(\mathbf{x})|}{r^N}=0. \tag{6.38}$$

From (6.37) and (6.38) we conclude that $\mathbf{x}\in E^0$ or $\mathbf{x}\in E^1$.

The previous analysis implies that

$$\partial^m E\setminus\partial^* E=\{\chi\in\mathbb{R}^N:\limsup_{r\to 0}\frac{\|D\chi_E\|(B_r(\mathbf{x}))}{r^{N-1}}>0\} \tag{6.39}$$

Moreover, we can write

$$\partial^m E \setminus \partial^* E = \cup_{k=1}^{\infty} A_k, \tag{6.40}$$

where

$$A_k = \{\chi \in \partial^m E \setminus \partial^* E : \limsup_{r\to 0} \frac{\|D\chi_E\|(B_r(\mathbf{x}))}{r^{N-1}} > \frac{1}{k}\}. \tag{6.41}$$

We now proceed to show that $\mathcal{H}^{N-1}(A_k) = 0$. We fix $k$ and let $\eta > 0$. Let $V$ be an open set such that $A_k \subset V$ and

$$\|D\chi_E\|(V) \le \|D\chi_E\|(A_k) + \eta \tag{6.42}$$

For any $\mathbf{x} \in A_k$, using (6.41), we can find $r_{\mathbf{x}} < \eta$ such that $\int_{B_r(\mathbf{x})} \|D\chi_E\| \ge \frac{r_{\mathbf{x}}^{N-1}}{2k}$ and $B_{r_{\mathbf{x}}}(\mathbf{x}) \subset V$. We have

$$A_k \subset \cup B_{r_{\mathbf{x}}}(\mathbf{x}) \tag{6.43}$$

A covering argument implies that there exists a countable collection $\{\mathbf{x}_i\} \subset A_k$ such that, setting $r_i := r_{\mathbf{x}_i}$,

$$A_k \subset B_{3r_i}(\mathbf{x}_i),\ \ B_{r_i}(\mathbf{x}_i) \cap B_{r_j}(\mathbf{x}_j) = \emptyset,\ i \ne j. \tag{6.44}$$

We can now compute

$$\begin{aligned}\sum_{i=1}^{\infty}(3r_i)^{N-1} &\le 3^{N-1}2k\,\|D\chi_E\|(B_{r_i(\mathbf{x}_i)})\\ &= 3^{N-1}2k\sum_{i=1}^{\infty}\|D\chi_E\|(B_{r_i(\mathbf{x}_i)})\\ &\le C\,\|D\chi_E\|(V)\\ &\le C(\|D\chi_E\|(A_k)+\eta)\end{aligned}$$

.

Therefore, since $\eta$ is arbitrary, we have

$$\mathcal{H}^{N-1}(A_k) \le C\,\|D\chi_E\|(A_k), \tag{6.45}$$

which implies that $\mathcal{H}^{N-1}(A_k) = 0$ because $\|D\chi_E\|(A_k) = \mathcal{H}^{N-1}(A_k \cap \partial^* E) = 0$. We conclude that $\mathcal{H}^{N-1}(\partial^m E \setminus \partial^* E) = 0$. □

We have:

**Theorem 6.6.** *Let $E$ be a set of finite perimeter. Then, $\partial^* E$ is a rectifiable set; that is,*

$$\partial^* E = (\cup C_i) \cup \mathcal{N},$$

*where $\mathcal{H}^{N-1}(\mathcal{N}) = 0$ and each $C_i$ is contained in a $C^1$ manifold $M_i$.*

*Proof.* Using Egorov's theorem, there exists, for each $i$, a set $F_i$ such that

$$\|\nabla\chi_E\|(\partial^* E \backslash F_i) < \frac{1}{2^i}$$

and the following limit is uniform for a.e. $\mathbf{x} \in \partial^* E$,

$$\lim_{r\to 0} \frac{|B_r(\mathbf{x}) \cap E \cap \pi_{-\nu}|}{r^N} = 0. \tag{6.46}$$

From Luzin's theorem, there exists a set $C_i$ such that

$$\|\nabla \chi_E\| (F_i \backslash C_i) < \frac{1}{2^i}$$

and

$$\nu \lfloor C_i \text{ is continuous.}$$

We now fix one of the sets $C_i$. Since $\nu \lfloor C_i$ is continuous it follows, from the proof of Lemma 6.2, that

$$|E \cap B_r(\mathbf{z})| \geq C r^N, \tag{6.47}$$

for all $\mathbf{z} \in C_i$ and $r$ small enough.

We now fix $\varepsilon > 0$. From (10.1) and (10.2) it follows that there exists $\sigma_\varepsilon > 0$ such that for all $r < 2\sigma_\varepsilon$ and $\mathbf{z} \in C_i$,

$$\frac{|E \cap B_r(\mathbf{z}) \cap \pi_{-\nu}|}{|B_r(\mathbf{z}) \cap \pi_{-\nu}|} < \frac{2\varepsilon^N C}{\omega_N 2^N} \tag{6.48}$$

and

$$|E \cap B_r(\mathbf{z})| \geq C r^N, \tag{6.49}$$

where $C$ is a universal constant.

We consider the cone $\mathcal{C}_\varepsilon$ with opening $2\theta$ such that $\cos\theta = \varepsilon$. We take $\mathbf{x} \in C_i$. We claim that

$$(\partial^* E \cap B_{\mathbf{x}}(\sigma_\varepsilon)) \subset \mathbf{x} + \mathcal{C}_\varepsilon.$$

Indeed, if this is not true, then there exists $by \in \partial^* E$, $|\mathbf{x} - \mathbf{y}| < \sigma_\varepsilon$ such that

$$B_{\varepsilon|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \cap \pi_{-\nu} = \phi \text{ or } B_{\varepsilon|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \cap \pi_{\nu} = \phi$$

Without loss of generality we can assume that

$$B_{\varepsilon|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \cap \pi_{\nu} = \phi \tag{6.50}$$

Using (10.3) we have

$$\begin{aligned}
|B_{\varepsilon|\mathbf{x}-\mathbf{y}|}(y) \cap E| &= |B_{\varepsilon|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \cap E \cap \pi_\nu| + |B_{\varepsilon|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \cap E \cap \pi_{-\nu}| \\
&= |B_{\varepsilon|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \cap E \cap \pi_{-\nu}| \\
&\leq |B_{2|\mathbf{x}-\mathbf{y}|}(\mathbf{x}) \cap E \cap \pi_{-\nu}| \\
&< \frac{2\varepsilon^N C}{w_N 2^N} \; |B_{2|\mathbf{x}-\mathbf{y}|} \cap \pi_{-\nu}| \\
&= \frac{2\varepsilon^N C}{w_N 2^N} \cdot \frac{2^N |\mathbf{x}-\mathbf{y}|^N w_N}{2} \\
&= C|\mathbf{x}-\mathbf{y}|^N \varepsilon^N
\end{aligned} \tag{6.51}$$

On the other hand, (10.4) implies that:

$$|B_{\varepsilon|\mathbf{x}-\mathbf{y}|}(\mathbf{y}) \cap E| \geq C\varepsilon^N |\mathbf{x}-\mathbf{y}|^N,$$

which contradicts (10.6).

Since $\varepsilon$ is arbitrary, we have proved that for any cone $\mathcal{C}$,

$$\partial^* E \cap B_r(\mathbf{x}) \subset \mathbf{x} + \mathcal{C},$$

for all $\mathbf{x} \in C_i$ and $r \leq r_0$ where $r_0$ depends on $\mathcal{C}$. This implies, see **??**, that $C_i$ is the zero level set of a $C^1$ function $f_i$, that is, $C_i$ is contained in the $C^1$ manifold $f_i^{-1}(0)$. □

We have the following

**Theorem 6.7.**

$$\|D\chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^* E \tag{6.52}$$

*Proof.* Let E be a set of finite perimeter and let $B \subset \partial^* E$. We first show that

$$\mathcal{H}^{N-1}(B) \leq C \|D\chi_E\| (B). \tag{6.53}$$

We let E,$\eta > 0$. We can find an open set.
A such that $B \subset A$ and

$$\|D\chi_E\| (A) \leq \|D\chi_E\| (B) + \eta \tag{6.54}$$

For any $\mathbf{x} \in B$, there exists $r_{\mathbf{x}}$ such that $r_{\mathbf{x}} < E$, $B_{r_{\mathbf{x}}}(\mathbf{x}) \subset A$ and

$$\|D\chi_E\| (B_{r_{\mathbf{x}}}(\mathbf{x})) \geq \frac{1}{2} w_{N-1} r_{\mathbf{x}}^{N-1}. \tag{6.55}$$

Thus, $\cup B_{r_{\mathbf{x}}}(\mathbf{x})$ covers $B$ and $a$ covering argument implies that there exist a countable collection $\mathbf{x}_i$ such that, if we denote $r_{\mathbf{x}_i} := r_i$, we have

$$B \subset \bigcup_{i=1}^{\infty} B_{3r_i}(\mathbf{x}_i) \textit{ and } B_{3r_i}(\mathbf{x}_i) \cap B_{3r_j}(\mathbf{x}_j) = \emptyset \tag{6.56}$$

We now compute, using (6.55),

$$\begin{aligned} \sum_{i=1}^{\infty} (3r_i)^{N-1} &\leq \frac{3^{N-1}2}{W_{N-1}} \sum_{i=1}^{\infty} \|D\chi_E\| B_{r_{\mathbf{x}}}(\mathbf{x}) && (6.57) \\ &\leq C \|D\chi_E\| (A) && (6.58) \\ &\leq C \left( \|D\chi_E\| (B) + \eta \right). && (6.59) \end{aligned}$$

Since Theorem 6.6, we know that $\partial^* E$ can be writen as

$$\partial^* E = (U_{i=1}^{\infty} C_i) \cup N, \tag{6.60}$$

where $\mathcal{H}^{N-1}(N) = 0$ and each $C_i$ is a compact set contained in a $C^1$ manifold, say $M_i$. We define

$$M_i := \mathcal{H}^{N-1} \llcorner M \tag{6.61}$$

If $\mathbf{x} \in C_i$, clearly

$$\lim_{r \to 0} \frac{M_i(B_r(\mathbf{x}))}{r^{N-1}} = w_{N-1} \tag{6.62}$$

On the other hand, since $\mathbf{x} \in \partial^* E$, we have

$$\lim_{r \to 0} \frac{\|D\chi_E\| (B_r(\mathbf{x}))}{r^{N-1}} = w_{N-1}. \tag{6.63}$$

From (6.62) and (6.63) we obtain

$$\lim_{r \to 0} \frac{M_i(B_r(\mathbf{x}))}{\|D\chi_E\| (B_r(\mathbf{x}))} = 1, \ \mathbf{x} \in C_i \tag{6.64}$$

□

**Theorem 6.8.** *Let $n > 1$ and $0 < \tau < \frac{1}{2}$. Suppose $E$ is a set of finite perimeter such that* $\limsup_{r\to 0} \frac{|E\cap B_r(x)|}{|B_r(x)|} > \tau$ *whenever $x \in E$. Then there exists a constant $C = C(r, n)$ and a sequence of closed balls $\overline{B}_{r_i}(x_i)$ with $x_i \in E$ such that*

$$E \subset \bigcup_{i=1}^{\infty} \overline{B}_{r_i}(x_i)$$

*and*

$$\sum_{i=1}^{\infty} r_i^{n-1} \leq C\mathcal{H}^{n-1}(\partial^* E).$$

*Remark* 6.9. If $E$ is an open set, the proof of Theorem 6.8 actually shows that we can take $\tau = \frac{1}{2}$. Moreover, the covering $\{B_{r_i}\}$ can be chosen in such a way that

$$\frac{\left|B_{r_i/5} \cap E\right|}{\left|B_{r_i/5}\right|} = \frac{1}{2}.$$

## 7. Almost one-sided smooth approximation of sets of finite perimeter

We now proceed to establish a fundamental approximation theorem for a set of finite perimeter by a family of sets with smooth boundary essentially from the measure-theoretic interior of the set with respect to any Radon measure that is absolutely continuous with respect to $\mathcal{H}^{N-1}$. That is, we prove that, for any Radon measure $\mu$ on $\mathbb{R}^N$ such that $\mu << \mathcal{H}^{N-1}$, the superlevel sets of the mollifications of the characteristic functions of sets of finite perimeter provide an approximation by smooth sets which are $\|\mu\|$-almost contained in the measure-theoretic interior of $E$.

It is a classical result in geometric measure theory that a set of finite perimeter $E$ can be approximated with smooth sets $E_k$ such that

$$\mathcal{L}^N(E_k) \to \mathcal{L}^N(E) \text{ and } P(E_k) \to P(E), \tag{7.1}$$

where $P(E)$ is the perimeter of $E$ and $\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$. The approximating smooth sets (see for instance Ambrosio-Fusco-Pallara [1, Remark 3.42] and Maggi [15, Theorem 13.8]) are the superlevel sets of the convolutions of $\chi_E$, which can be chosen for a.e. $t \in (0, 1)$. The one-sided approximation refines the classical result in the sense that it distinguishes between the superlevel sets for a.e. $t \in (\frac{1}{2}, 1)$ from the ones corresponding to a.e. $t \in (0, \frac{1}{2})$, thus providing an *interior* and an *exterior* approximation of the set respectively (see Theorem 8.1 and Theorem 9.1). Indeed, in the first case, the difference between the level sets and the measure theoretic interior is asymptotically vanishing with respect to the $\mathcal{H}^{N-1}$-measure; in the latter, we obtain the same result for the measure theoretic exterior.

## 8. The approximation of $E$ with respect to any $\mu << \mathcal{H}^{N-1}$

The one-sided approximation theorem allows to extend (7.1) to any Radon measure $\mu$ such that $\mu << \mathcal{H}^{N-1}$. More precisely, for any bounded set of finite perimeter $E$, there exist smooth sets $E_{k;i}$, $E_{k;e}$, such that

$$\mu(E_{k;i}) \to \mu(E^1), \quad P(E_{k;i}) \to P(E) \tag{8.1}$$

and

$$\mu(E_{k;e}) \to \mu(E), \quad P(E_{k;e}) \to P(E). \tag{8.2}$$

The convergence of the perimeters in (8.1) and (8.2) follows as in the standard proof of (7.1). However, the convergence with respect to $\mu$ is a consequence of the following result.

**Theorem 8.1.** *Let $\mu$ be a Radon measure such that $\mu << \mathcal{H}^{N-1}$ and $E$ be a bounded set of finite perimeter in $\mathbb{R}^N$. Then:*
**(a)** $\|\mu\|(E^1\Delta A_{k;t}) \to 0$*, for* $\frac{1}{2} < t < 1$*;*
**(b)** $\|\mu\|(E\Delta A_{k;t}) \to 0$*, for* $0 < t < \frac{1}{2}$*.*

*Proof.* We have

$$u_k(x) \to u_E(x) \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x. \tag{8.3}$$

Since $\{0 < |u_k - u_E| \leq 1\} \subset E_\delta := \{x \in \mathbb{R}^N : \text{dist}(x, E) \leq \delta\}$, for any $k$ if $\delta > \max \varepsilon_k$, and $E_\delta$ is bounded, then we can apply the dominated convergence theorem with respect to the measure $\|\mu\|$, taking 1 as summable majorant since $\mu$ is a Radon measure. Hence, for any $\varepsilon > 0$, there exists $k$ large enough such that, if $\frac{1}{2} < t < 1$, we have

$$\begin{aligned}
\varepsilon &\geq \int_{\mathbb{R}^n} |u_k(x) - u_E(x)| d\,\|\mu\| \\
&\geq \int_{A_{k;t}\setminus E^1} |u_k(x) - u_E(x)| d\,\|\mu\| + \int_{E^1\setminus A_{k;t}} |u_E(x) - u_k(x)| d\,\|\mu\| \\
&\geq (t - \frac{1}{2})\,\|\mu\|\,(A_{k;t} \setminus E^1) + (1-t)\,\|\mu\|\,(E^1 \setminus A_{k;t}) \\
&\geq \min\left\{t - \frac{1}{2}, 1 - t\right\} \|\mu\|\,(A_{k;t}\Delta E^1).
\end{aligned}$$

Thus, for $k$ large enough and $\frac{1}{2} < t < 1$, we obtain

$$\|\mu\|\,(A_{k;t}\Delta E^1) \leq \frac{\varepsilon}{\min\left\{t - \frac{1}{2}, 1 - t\right\}},$$

which is (a). Analogously, for $0 < t < \frac{1}{2}$, we have

$$\begin{aligned}
\varepsilon &\geq \int_{\mathbb{R}^n} |u_k(x) - u_E(x)| d\,\|\mu\| \\
&\geq \int_{A_{k;t}\setminus E} |u_k(x) - u_E(x)| d\,\|\mu\| + \int_{E\setminus A_{k;t}} |u_E(x) - u_k(x)| d\,\|\mu\| \\
&\geq t\,\|\mu\|\,(A_{k;t} \setminus E) + (\frac{1}{2} - t)\,\|\mu\|\,(E \setminus A_{k;t}) \\
&\geq \min\left\{t, \frac{1}{2} - t\right\} \|\mu\|\,(A_{k;t}\Delta E).
\end{aligned}$$

Thus, for large $k$ and $0 < t < \frac{1}{2}$,

$$\|\mu\|\,(A_{k;t}\Delta E) \leq \frac{\varepsilon}{\min\left\{t, \frac{1}{2} - t\right\}},$$

which gives (b). □

*Remark* 8.2. The convergence in (8.1) follows easily from Theorem 8.1: we have

$$|\mu(E^1) - \mu(A_{k;t})| = |\mu(E^1 \setminus A_{k;t}) - \mu(A_{k;t} \setminus E^1)|$$

and it is clear that (a) implies

$$|\mu(E^1 \setminus A_{k;t})| \le \|\mu\| (E^1 \setminus A_{k;t}) \to 0,$$
$$|\mu(A_{k;t} \setminus E^1)| \le \|\mu\| (A_{k;t} \setminus E^1) \to 0.$$

One can show (8.2) in a similar way using (b).
We also notice that Theorem 8.1 has been proved for any $t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. However, since the sets $A_{k;t}$ have smooth boundary only for almost every $t$, we shall consider only $t \notin Z$, where $Z$ is the set of singular values defined in the introduction.

*Remark* 8.3. With $\mu = \mathcal{H}^{N-1} \llcorner \partial^* E$, we obtain from Theorem 8.1:
**(a)** $\mathcal{H}^{N-1}(\partial^* E \cap A_{k;t}) \to 0$ for $\frac{1}{2} < t < 1$;
**(b)** $\mathcal{H}^{N-1}(\partial^* E \cap (\mathbb{R}^N \setminus A_{k;t})) \to 0$ for $0 < t < \frac{1}{2}$.
Indeed, this is clear from the following identities

$$\partial^* E \cap (E^1 \Delta A_{k;t}) = \partial^* E \cap [(E^1 \setminus A_{k;t}) \cup (A_{k;t} \setminus E^1)] = \partial^* E \cap A_{k;t},$$

$$\partial^* E \cap (E \Delta A_{k;t}) = \partial^* E \cap [(E \setminus A_{k;t}) \cup (A_{k;t} \setminus E)] = \partial^* E \cap (\mathbb{R}^N \setminus A_{k;t}).$$

*Remark* 8.4. Using Remark 8.3 we can also show that we have:
**(a)** $\mathcal{H}^{N-1}(\partial^* E \cap u_k^{-1}(t)) \to 0$ for $\frac{1}{2} < t < 1$;
**(b)** $\mathcal{H}^{N-1}(\partial^* E \cap u_k^{-1}(t)) \to 0$ for $0 < t < \frac{1}{2}$.
Indeed, $u_k^{-1}(t) \subset A_{k;s}$ for $\frac{1}{2} < s < t < 1$ and $u_k^{-1}(t) \subset \mathbb{R}^N \setminus A_{k;s}$ for $0 < t \le s < \frac{1}{2}$.

In addition, we observe that $\|\mu\| (u_k^{-1}(t)) = 0$ for $\mathcal{L}^1$-a.e. $t$, since $\mu$ is a Radon measure. It is in fact clear that $u_k^{-1}(t) = \partial A_{k;t}$, that $\overline{A_{k;t}} \subset A_{k;s} \subset A_{k;0}$ if $0 < s < t < 1$, with $A_{k;0}$ bounded, and that the sets $\partial A_{k;t}$ are pairwise disjoint. Hence, since $\|\mu\|$ is finite on bounded sets and additive, the set

$$\{t \in (0,1) : \|\mu\| (\partial A_{k;t}) > \varepsilon\}$$

is finite for any $\varepsilon > 0$. This implies that the set $\{t \in (0,1) : \|\mu\| (\partial A_{k;t}) > 0\}$ is at most countable (see also the observation at the end of Section 1.4 of [1]).

Then we obtain also:
**(a)** $\mathcal{H}^{N-1}(\partial^* E \cap u_k^{-1}(t)) = 0$ for a.e. $\frac{1}{2} < t < 1$;
**(b)** $\mathcal{H}^{N-1}(\partial^* E \cap u_k^{-1}(t)) = 0$ for a.e. $0 < t < \frac{1}{2}$.

## 9. The main approximation result

The following theorem, together with Theorem 8.1, shows that indeed we have an *interior* approximation of $E$ for a.e. $t \in (\frac{1}{2}, 1)$.

**Theorem 9.1.** *Let $E$ be a set of finite perimeter in $\mathbb{R}^N$. There exists a sequence $\varepsilon_k$ converging to 0 such that, if $u_k := \chi_E * \rho_{\varepsilon_k}$, we have*

$$\lim_{k \to +\infty} \mathcal{H}^{N-1}(u_k^{-1}(t) \setminus E^1) = 0 \tag{9.1}$$

*for a.e. $t \in (\frac{1}{2}, 1)$.*

*Proof.* We take $s > \frac{1}{2}$ and a sequence $\varepsilon_k$, with $\varepsilon_k \to 0$, and we consider the set $A_{k;s} := \{u_k > s\}$. By the coarea formula (5.4), we have

$$\begin{aligned}\int_{A_{k;s}\setminus E^1} |\nabla u_k|\,dx &= \int_0^1 \mathcal{H}^{N-1}(u_k^{-1}(t)\cap(A_{k;s}\setminus E^1))\,dt \\ &= \int_s^1 \mathcal{H}^{N-1}(u_k^{-1}(t)\setminus E^1)\,dt,\end{aligned} \tag{9.2}$$

since, for $t \le s$, $u_k^{-1}(t)\cap(A_{k;s}\setminus E^1) = \emptyset$, while, for $t > s$, $u_k^{-1}(t)\cap(A_{k;s}\setminus E^1) = u_k^{-1}(t)\setminus E^1$.

We claim that

$$\|\nabla u_k\|_{L^1(A_{k;s}\setminus E^1)} \to 0. \tag{9.3}$$

In order to prove the claim, we observe that, for any $x \in \mathbb{R}^N$,

$$\begin{aligned}\nabla u_k(x) = \int_{\mathbb{R}^N} \chi_E(y)\nabla_x\rho_{\varepsilon_k}(x-y)\,dy &= -\int_{\mathbb{R}^N} \chi_E(y)\nabla_y\rho_{\varepsilon_k}(x-y)\,dy \\ &= \int_{\mathbb{R}^N} \rho_{\varepsilon_k}(x-y)\nu_E(y)\,d\,\|D\chi_E\|\,(y) = (\rho_{\varepsilon_k} * D\chi_E)(x).\end{aligned}$$

Hence, $\nabla u_k = (D\chi_E * \rho_{\varepsilon_k}) = (\|D\chi_E\|\,\nu_E * \rho_{\varepsilon_k})$, which implies

$$|\nabla u_k| \le \|D\chi_E\| * \rho_{\varepsilon_k}. \tag{9.4}$$

Recalling from (5.1) that $\mathcal{L}^N(E\Delta E^1) = 0$, (9.4) leads to

$$\begin{aligned}\|\nabla u_k\|_{L^1(A_{k;s}\setminus E^1)} &= \int_{\mathbb{R}^N} |\nabla u_k|\,\chi_{A_{k;s}\setminus E}\,dx \\ &\le \int_{\mathbb{R}^N} (\|D\chi_E\| * \rho_{\varepsilon_k})\,\chi_{A_{k;s}\setminus E}\,dx = \int_{\mathbb{R}^N} (\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})\,d\,\|D\chi_E\| = \\ &= \int_{\partial^* E} (\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})\,d\mathcal{H}^{N-1}.\end{aligned}$$

Thus, we need to investigate, for any $x \in \partial^* E$, the behaviour of $(\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})(x)$ as $k \to +\infty$. We have

$$\begin{aligned}(\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})(x) &= \int_{\mathbb{R}^N} \varepsilon_k^{-N}\rho\left(\frac{x-y}{\varepsilon_k}\right)\,\chi_{A_{k;s}}(y)\,\chi_{(\mathbb{R}^N\setminus E)}(y)\,dy \\ &= [y = x+\varepsilon_k z] = \int_{B_1(0)} \rho(z)\,\chi_{A_{k;s}}(x+\varepsilon_k z)\,\chi_{(\mathbb{R}^N\setminus E)}(x+\varepsilon_k z)\,dz.\end{aligned}$$

We observe that $x+\varepsilon_k z \in \mathbb{R}^N\setminus E$ if and only if $z \in \frac{(\mathbb{R}^N\setminus E)-x}{\varepsilon_k}$, hence

$$\chi_{(\mathbb{R}^N\setminus E)}(x+\varepsilon_k\cdot) = \chi_{\frac{(\mathbb{R}^N\setminus E)-x}{\varepsilon_k}}(\cdot) \to \chi_{H^-_{\nu_E(x)}}(\cdot) \quad \text{in} \quad L^1(B_1(0)) \quad \text{as} \quad k \to +\infty.$$

In particular, this means that the $L^1$ limit of $\chi_{(\mathbb{R}^N\setminus E)}(x+\varepsilon_k z)$ is not $\mathcal{L}^N$-a.e. zero only if $z\cdot\nu_E(x) \le 0$, so we can restrict the integration domain to $B_1(0)\cap H^-_{\nu_E(x)}$. On the other hand, $x+\varepsilon_k z \in A_{k;s} = \{u_k > s\}$ if and only if $u_k(x+\varepsilon_k z) > s$. We see that

$$\begin{aligned}u_k(x+\varepsilon_k z) &= \int_{\mathbb{R}^N} \rho_{\varepsilon_k}(x+\varepsilon_k z - y)\,\chi_E(y)\,dy \\ &= [y = x+\varepsilon_k z+\varepsilon_k u] = \int_{B_1(0)} \rho(u)\,\chi_E(x+\varepsilon_k(u+z))\,du.\end{aligned}$$

Arguing as before, we obtain $\chi_E(x+\varepsilon_k(z+\cdot)) \to \chi_{H^+_{\nu_E}(x)}(z+\cdot)$ in $L^1(B_1(0))$ as $k \to +\infty$, for any $x \in \partial^* E$ and $z \in B_1(0)$. Now, we recall that $z \cdot \nu_E(x) \leq 0$, and, since we have $\chi_{H^+_{\nu_E}(x)}(z+u) = 1$ if and only if $0 \leq (z+u)\cdot \nu_E(x)$, we conclude that $0 \leq -z \cdot \nu_E(x) \leq u \cdot \nu_E(x) \leq 1$; that is, $u$ belongs to the half ball $B_1(0) \cap H^+_{\nu_E}(x)$. This implies that, for any $x \in \partial^* E$ and $z \in B_1(0) \cap H^-_{\nu_E}(x)$,

$$\lim_{k\to+\infty} u_k(x+\varepsilon_k z) := v(x,z) = \int_{B_1(0)} \rho(u)\, \chi_{H^+_{\nu_E}(x)}(z+u)\, du \leq \frac{1}{2}. \tag{9.5}$$

Therefore, these calculations yield

$$\begin{aligned} (\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E})(x) &= \int_{B_1(0)} \rho(z)\, \chi_{A_{k;s}}(x+\varepsilon_k z)\, \chi_{(\mathbb{R}^N\setminus E)}(x+\varepsilon_k z)\, dz \\ &\to \int_{B_1(0)} \rho(z)\, \chi_{\{v(x,z)>s\}}(z)\, \chi_{H^-_{\nu_E}(x)}(z)\, dz, \end{aligned} \tag{9.6}$$

for any $x \in \partial^* E$.

Equation (9.5) shows then that the limit in (9.6) is identically zero, since

$$\left\{ z \in \mathbb{R}^N : v(x,z) > s > \frac{1}{2} \right\} \cap B_1(0) \cap H^-_{\nu_E}(x) = \emptyset,$$

for any $x \in \partial^* E$.

We can now apply the Lebesgue dominated convergence theorem with respect to the measure $\mathcal{H}^{N-1} \llcorner \partial^* E$ and the sequence of functions $\rho_{\varepsilon_k} * \chi_{A_{k;s}\setminus E}$ (since the constant 1 is clearly a summable majorant), thus obtaining (9.3).

Finally, up to passing to a subsequence (which we shall keep calling $\varepsilon_k$ with a little abuse of notation), (9.2) and (9.3) yield (9.1), for a.e. $t > s$. Since $s > \frac{1}{2}$ is fixed arbitrarily, we can conclude that (9.1) is valid for a.e. $t > \frac{1}{2}$.

□

For the case $t = \frac{1}{2}$ we have:

**Lemma 9.2.** *Let $\mu$ be a Radon measure on $\mathbb{R}^N$ such that $\mu << \mathcal{H}^{N-1}$. Let $E$ be a set of finite perimeter, and let $u_k$ be the mollification of $\chi_E$. Then, for $t = \frac{1}{2}$ and $\varepsilon > 0$, there exists $k^* = k^*(\varepsilon)$ such that*

$$\|\mu\|(E^1 \setminus A_{k;\frac{1}{2}}) < \varepsilon \quad \text{and} \quad \|\mu\|(A_{k;\frac{1}{2}} \setminus E) < \varepsilon. \tag{9.7}$$

*Proof.* Since $u_k(y) \to u_E(y)$ for $\mathcal{H}^{N-1}$-a.e. $y$, the dominated convergence theorem implies that $u_k \to u_E$ in $L^1(\mathbb{R}^N, \|\mu\|)$. Thus, given any $\varepsilon > 0$, for $k$ large enough, we have

$$\frac{\varepsilon}{2} \geq \int_{\mathbb{R}^N} |u_E - u_k|\, d\|\mu\| \geq \int_{E^1\setminus A_{k;\frac{1}{2}}} (u_E - u_k) d\|\mu\| \geq (1-\frac{1}{2})\|\mu\|(E^1 \setminus A_{k;\frac{1}{2}}), \tag{9.8}$$

which implies

$$\|\mu\|(E^1 \setminus A_{k;\frac{1}{2}}) \leq \varepsilon. \tag{9.9}$$

In the same way, we compute

$$\frac{\varepsilon}{2} \geq \int_{A_{k;\frac{1}{2}}\setminus E} |u_E - u_k|\, d\|\mu\| \geq (\frac{1}{2} - 0)\|\mu\|(A_{k;\frac{1}{2}} \setminus E), \tag{9.10}$$

which implies

$$\|\mu\|(A_{k;\frac{1}{2}} \setminus E) \leq \varepsilon. \tag{9.11}$$

□

The following remark shows that, with $t = \frac{1}{2}$ and with $\mu = \mathcal{H}^{N-1} \llcorner \partial^* E \geq 0$, parts of Theorem 8.1 do not hold.

*Remark* 9.3. If we define $E := \{y \in \mathbb{R}^N : |y| \leq 1\}$, then $u_k^{-1}(\frac{1}{2}) \subset \mathbb{R}^N \setminus E$ for all $k$, and therefore it is clear that

$$\mathcal{H}^{N-1}((A_{k;\frac{1}{2}} \setminus E^1) \cap \partial^* E) = \mathcal{H}^{N-1}(\partial^* E) \nrightarrow 0 \quad \text{as } k \to \infty. \tag{9.12}$$

If we now define $E := \{y \in \mathbb{R}^N : |y| \geq 1\}$, then $u_k^{-1}(\frac{1}{2}) \subset E$ for all $k$ and thus

$$\mathcal{H}^{N-1}((E \setminus A_{k;\frac{1}{2}}) \cap \partial^* E) = \mathcal{H}^{N-1}(\partial^* E) \nrightarrow 0 \quad \text{as } k \to \infty. \tag{9.13}$$

**Approximation of sets of finite perimeter completely from the inside.** It is well-known that, a set of finite perimeter, $E$, cannot be approximated by smooth sets that lie completely in the interior of $E$. For example, consider the open unit disk with a single radius removed, and let $U$ be the resulting open set. Then the Hausdorff measure of the boundary of $U$ is $2\pi$ plus the measure of the radius, while the Hausdorff measure of the reduced boundary is $2\pi$. Thus, if $U_k$ is an approximating open subset of $U$, then its boundary will be close to that of boundary $U$ and so its the Hausdorff measure will be close to $2\pi$ plus 1. Adding more radii, say $m$ of them, will force the approximating set to have boundaries whose Hausdorff measure close to $2\pi$ plus $m$. In general, if we let $K$ denote any compact subset without interior and of infinite Hausdorff measure, then the approximating sets will have boundaries whose measures will necessarily tend to infinity.

We have the following:

**Proposition 1.** *Let $U \subset \mathbb{R}^N$ be an open set with $\mathcal{H}^{N-1}(\partial U) < \infty$. Then there exists a sequence of bounded open sets $U_k \subset \overline{U_k} \subset U$ such that*

(i) $|U_k| = |\overline{U_k}|$*;*
(ii) $|U_k| \to |U|$*;*
(iii) $\mathcal{H}^{N-1}(\partial U_k) \to \mathcal{H}^{N-1}(\partial U)$*.*

*Proof.* By definition, for each integer $k$, there exists a covering of $\partial U$ by balls

$$\partial U \subset \bigcup B_i(r_i),$$

each with radius $r_i$, such that

$$\sum_{i=1}^{\infty} \mathcal{H}^{N-1}\big(\partial B_i(r_i)\big) = \sum_{i=1}^{\infty} \omega_{N-1} r_i^{N-1} < \mathcal{H}^{N-1}(\partial U) + \frac{1}{k},$$

where $\omega_{N-1}$ is the $\mathcal{H}^{N-1}$ measure of the boundary of the unit ball in $\mathbb{R}^N$. Since $\partial U$ is compact, the covering may be taken as a finite covering, say by $m$ of them, $B_1(r_1), B_1(r_2), \ldots, B_m(r_m)$. Then the open set $V_k := \bigcup B_i(r_i)$ has the property that

$$\partial V_k \subset \bigcup_{i=1}^{m} \partial B_i(r_i)$$

and therefore that

$$\mathcal{H}^{N-1}(\partial V_k) \leq \mathcal{H}^{N-1}(\bigcup_{i=1}^{m} \partial B_i(r_i)) \leq \sum_{i=1}^{\infty} \omega_{N-1} r_i^{N-1} < \mathcal{H}^{N-1}(\partial U) + \frac{1}{k}.$$

Thus, the open sets $U_k := U \setminus \overline{V}_k \subset U$ will satisfy our desired result, except that they are not smooth. □

Given an arbitrary set of finite perimeter, $E$, we know from §4 that $E$ can be approximated by sets with smooth boundaries essentially from the measure-theoretic interior of $E$, that is, a one-sided approximation can "almost" be achieved (see Theorem 7.1**(e)**). On the other hand, the next result shows that, if $E$ is sufficiently regular, there does, in fact, exist a one-sided approximation. The condition of regularity we impose is similar to Lewis's *uniformly flat* condition in potential theory [**?**].

**Theorem 9.4.** *Suppose that $E$ is a bounded set of finite perimeter with the property that, for all $y \in \partial E$, there are positive constants $c_0$ and $r_0$ such that*

$$\frac{\left|E^0 \cap B(y,r)\right|}{|B(y,r)|} \geq c_0 \qquad \text{for all } r \leq r_0. \tag{9.14}$$

*Then there exists $t \in (0,1)$ such that*

$$A_{k;t} \Subset E \qquad \text{for large } k. \tag{9.15}$$

*Proof.* Choose a mollifying kernel $\rho$ such that $\rho = 1$ on $B(0, \frac{1}{2})$. If $y \in \partial E$, we have

$$\begin{aligned} v_k(y) := \chi_{\mathbb{R}^N \setminus E} * \rho_{\varepsilon_k}(y) &= \frac{1}{\varepsilon_k^N} \int_{B(y,\varepsilon_k)} \chi_{\mathbb{R}^N \setminus E}(x) \rho\Big(\frac{x-y}{\varepsilon_k}\Big)\, dx \\ &\geq \frac{1}{\varepsilon_k^N} \int_{B(y,\frac{\varepsilon_k}{2})} \chi_{\mathbb{R}^N \setminus E}(x)\, dx \\ &= \frac{|(\mathbb{R}^N \setminus E) \cap B(y, \frac{\varepsilon_k}{2})|}{\varepsilon_k^N} \\ &= \frac{|E^0 \cap B(y, \frac{\varepsilon_k}{2})|}{\varepsilon_k^N} \geq c_0/2^N := \tilde{c_0}, \end{aligned}$$

where $0 < \tilde{c_0} < 1$ depends only on the dimension $N$ and is independent of the point $y$. Note that $u_k(y) + v_k(y) = 1$ for all $y \in \mathbb{R}^N$. Therefore, for all $y \in \partial E$,

$$u_k(y) = 1 - v_k(y) \leq 1 - \tilde{c_0}.$$

Thus, taking $1 - \tilde{c_0} < t < 1$, we see that $A_{k;t} \cap \partial E = \emptyset$. Consequently, each connected component of the open set $A_{k;t}$ lies either in the interior of $E$ or in its exterior, and thus must lie in its interior. □

**Corollary 9.5.** *Let $E$ be a bounded set of finite perimeter with uniform Lipschitz boundary. Then there exists $T \in (0,1)$ such that $A_{k;T} \Subset E$.*

*Proof.* Since $E$ has a uniform Lipschitz boundary, for each $x \in \partial E$, there is a finite cone, $C_x$, with vertex $x$ that completely lies in the complement of $E$. Each cone $C_x$ is assumed to be congruent to a fixed cone $C$. This implies that the hypothesis of Theorem 9.4 is satisfied. Therefore, there exists $0 < T < 1$ such that $u_k(y) < T$ for all $k$ and all $y \in \partial E$. □

## 10. Interior approximations of open sets with $C^0$ and Lipschitz boundary.

These will be discussed in class and used to obtain Generalized Gauss-Green formulas.

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