Solutions to exam problems.

1. Recall that time-dependent potential flow system in n dimensions is:

$$\begin{aligned} &\hat\rho_t + \mathrm{div}\,(\hat\rho\,\nabla\Phi) = 0, \\ &\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + \frac{\hat\rho^{\gamma-1}-1}{\gamma-1} = K, \end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \mathbf{R}^n$, $t \in \mathbf{R}^1$, and Φ and $\hat\rho$ are functions of $(t, \boldsymbol{x})$, and $\gamma > 1$, $K > -\frac{1}{\gamma-1}$ are a given constants. Derive self-similar potential flow equation. That is, show that if a solution of (1) has the form $\Phi(t,\boldsymbol{x}) = t\psi(\frac{\boldsymbol{x}}{t})$, $\hat\rho(t,\boldsymbol{x}) = \rho(\frac{\boldsymbol{x}}{t})$, then the functions $\varphi(\boldsymbol{\xi}) = -\frac{|\boldsymbol{\xi}|^2}{2} + \psi(\boldsymbol{\xi})$ and $\rho(\boldsymbol{\xi})$ (where $\boldsymbol{\xi} \in \mathbf{R}^n$) satisfy:

$$\mathrm{div}\,\big(\rho(|\nabla\varphi|^2,\varphi)\nabla\varphi\big) + n\rho(|\nabla\varphi|^2,\varphi) = 0 \tag{2}$$

with

$$\rho(|\nabla\varphi|^2,\varphi) = \big(\rho_0^{\gamma-1} - (\gamma-1)(\varphi + \frac{1}{2}|\nabla\varphi|^2)\big)^{\frac{1}{\gamma-1}}, \tag{3}$$

where ρ_0 is a constant.

Solution. Differentiating, and denoting $\boldsymbol{\xi} = \frac{\boldsymbol{x}}{t}$, we obtain:

$$\begin{aligned} &\partial_t\Phi(t,\boldsymbol{x}) = \partial_t\big(t\psi(\frac{\boldsymbol{x}}{t})\big) = \psi(\frac{\boldsymbol{x}}{t}) - \frac{1}{t}\,\boldsymbol{x}\cdot\nabla_{\boldsymbol{\xi}}\psi(\frac{\boldsymbol{x}}{t}) = \psi(\boldsymbol{\xi}) - \boldsymbol{\xi}\cdot\nabla\psi(\boldsymbol{\xi}); \\ &\nabla_{\boldsymbol{x}}\Phi(t,\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\big(t\psi(\frac{\boldsymbol{x}}{t})\big) = \nabla_{\boldsymbol{\xi}}\psi(\frac{\boldsymbol{x}}{t}) = \nabla\psi(\boldsymbol{\xi}); \\ &\partial_t\hat\rho(t,\boldsymbol{x}) = \partial_t\big(\rho(\frac{\boldsymbol{x}}{t})\big) = -\frac{1}{t^2}\boldsymbol{x}\cdot\nabla_{\boldsymbol{\xi}}\rho(\frac{\boldsymbol{x}}{t}) = -\frac{1}{t}\,\boldsymbol{\xi}\cdot\nabla\rho(\boldsymbol{\xi}). \end{aligned}$$

From this, noting that $\nabla_{\boldsymbol{x}}(g(\frac{\boldsymbol{x}}{t})) = \frac{1}{t}\nabla_{\boldsymbol{\xi}}g(\boldsymbol{\xi})$ for any $g(\boldsymbol{\xi})$, we get

$$\mathrm{div}_{\boldsymbol{x}}(\hat\rho\,\nabla\Phi)(\boldsymbol{x},t) = \nabla_{\boldsymbol{x}}\cdot(\hat\rho\,\nabla\Phi)(\boldsymbol{x},t) = \frac{1}{t}\nabla_{\boldsymbol{\xi}}\cdot(\rho\nabla\psi)(\boldsymbol{\xi}) = \frac{1}{t}\mathrm{div}_{\boldsymbol{\xi}}(\rho\nabla\psi)(\boldsymbol{\xi}).$$

Substitute all expressions obtained above into (1), get, in $\boldsymbol{\xi}$-variables,

$$\begin{aligned} &-\boldsymbol{\xi}\cdot\nabla\rho + \mathrm{div}\,(\rho\nabla\psi) = 0, \\ &\psi - \boldsymbol{\xi}\cdot\nabla\psi + \frac{1}{2}|\nabla\psi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1}, \end{aligned} \tag{4}$$

where $\rho_0 > 0$ is determined from $\frac{\rho_0^{\gamma-1}}{\gamma-1} = K + \frac{1}{\gamma-1}$, and we use that $K + \frac{1}{\gamma-1} > 0$.

Now we have $\psi(\boldsymbol{\xi}) = \frac{|\boldsymbol{\xi}|^2}{2} + \varphi(\boldsymbol{\xi})$, so $\nabla\psi = \boldsymbol{\xi} + \nabla\varphi$. Substitute this to the second equation of (4), get

$$\frac{|\boldsymbol{\xi}|^2}{2} + \varphi - \boldsymbol{\xi}\cdot(\boldsymbol{\xi}+\nabla\varphi) + \frac{1}{2}|\boldsymbol{\xi}+\nabla\varphi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1},$$

which is

$$\varphi + \frac{1}{2}|\nabla\varphi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1}.$$

The last equation implies (3).

Similarly, from the first equation of (4), we get

$$-\boldsymbol{\xi}\cdot\nabla\rho + \operatorname{div}\left(\rho(\boldsymbol{\xi}+\nabla\varphi)\right) = 0. \tag{5}$$

Now we note that

$$\operatorname{div}(\rho\boldsymbol{\xi}) = \rho\operatorname{div}\boldsymbol{\xi} + \nabla\rho\cdot\boldsymbol{\xi} = n\rho + \boldsymbol{\xi}\cdot\nabla\rho, \tag{6}$$

where we used that in n dimensions, $\operatorname{div}\boldsymbol{\xi} = \sum_{i=1}^{n}\frac{\partial\xi_i}{\partial\xi_i} = n$. Substituting (6) into (5), and recalling (3), we obtain (2).

2. Let $\Omega \in \mathbf{R}^n$ be an open bounded set with smooth boundary, and let Γ be a relatively open subset of $\partial\Omega$. Assume that $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x})u_{x_ix_j} + \sum_{i=1}^{n} b_i(\boldsymbol{x})u_{x_i} = 0 \quad \text{in } \Omega,$$

$$\sum_{i=1}^{n}\beta_i(\boldsymbol{x})u_{x_i} = 0 \quad \text{on } \Gamma,$$

where $a_{ij}, b_i \in C(\overline{\Omega})$, $\beta_i \in C(\overline{\Gamma})$, and equation is elliptic in $\overline{\Omega}$, and the boundary condition is oblique on $\overline{\Gamma}$, that is, there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x})\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for all } \boldsymbol{x}\in\overline{\Omega},\ \ \xi = (\xi_1,\dots,\xi_n)\in\mathbf{R}^n;$$

$$\sum_{i=1}^{n}\beta_i(\boldsymbol{x})\nu_i(\boldsymbol{x}) \geq \lambda \quad \text{for all } \boldsymbol{x}\in\overline{\Gamma},$$

where $\boldsymbol{\nu}(\boldsymbol{x}) = (\nu_1(\boldsymbol{x}),\dots,\nu_n(\boldsymbol{x}))$ is the interior unit normal to $\partial\Omega$ at $\boldsymbol{x}\in\partial\Omega$.

Let $M = \max_{\boldsymbol{x}\in\overline{\Omega}} u(\boldsymbol{x})$, and $m = \min_{\boldsymbol{x}\in\overline{\Omega}} u(\boldsymbol{x})$. Assume $m < M$, i.e. u is non-constant. Prove that

$$m < u(\boldsymbol{x}) < M \quad \text{for all } \boldsymbol{x}\in\Omega\cup\Gamma,$$

i.e. that minimum and maximum cannot be attained on $\Omega\cup\Gamma$.

Solution. Extrema of u cannot be attained in Ω by the strong maximum principle. It remains to show that extrema cannot be attained on Γ.

Suppose $\boldsymbol{x}_0 \in \Gamma$, and maximum of u is attained at $\boldsymbol{x}_0$, i.e. $u(\boldsymbol{x}_0) = M$. Then, using that Γ is relatively open in $\partial\Omega$, we obtain

$$\boldsymbol{\tau} \cdot \nabla u = 0 \tag{7}$$

for any vector $\boldsymbol{\tau}$ tangential to $\partial\Omega$ at $\boldsymbol{x}_0$.

Let $\beta = (\beta_1, \ldots, \beta_n)$. Then boundary condition on Γ can be written as

$$\beta \cdot \nabla u = 0 \quad \text{on } \Gamma. \tag{8}$$

Denote $\boldsymbol{\tau} = \beta - (\beta \cdot \boldsymbol{\nu})\boldsymbol{\nu}$. Then (8) becomes

$$(\beta \cdot \boldsymbol{\nu}) u_{\boldsymbol{\nu}} + \boldsymbol{\tau} \cdot \nabla u = 0 \quad \text{on } \Gamma. \tag{9}$$

From now on all calculations are at point $\boldsymbol{x}_0$, so we write $\boldsymbol{\nu}$, β for $\boldsymbol{\nu}(\boldsymbol{x}_0)$, $\beta(\boldsymbol{x}_0)$. Since the vector $\boldsymbol{\tau} = \beta - (\beta \cdot \boldsymbol{\nu})\boldsymbol{\nu}$ is orthogonal to $\boldsymbol{\nu}$ (which checked explicitly, using $|\boldsymbol{\nu}| = 1$), and thus tangential to $\partial\Omega$ at $\boldsymbol{x}_0$, we have from (9) and (7)

$$(\beta \cdot \boldsymbol{\nu}) u_{\boldsymbol{\nu}} = 0 \quad \text{at } \boldsymbol{x}_0.$$

Now we note that obliqueness condition can be written as $\beta \cdot \boldsymbol{\nu} \geq \lambda > 0$ on Γ. Thus,

$$u_{\boldsymbol{\nu}} = 0 \quad \text{at } \boldsymbol{x}_0.$$

However, since x_0 is a point of maximum of u, and u is non-constant, Hopf's lemma implies $u_{\boldsymbol{\nu}} < 0$. Thus we arrived at a contradiction. This shows that maximum of u cannot be attained on Γ. Argument for the minimum is similar.

3. Let $M > 0$, and let $Cone \subset \mathbf{R}^{n+1}$ be the set defined by

$$Cone = \{(\boldsymbol{x}, x_{n+1}) \mid \boldsymbol{x} \in \mathbf{R}^n,\ x_{n+1} \in \mathbf{R} \text{ satisfying } x_{n+1} > M|\boldsymbol{x}|\}.$$

Also, for $\boldsymbol{b} \in \mathbf{R}^n$, $b_{n+1} \in \mathbf{R}$, we denote

$$Cone + (\boldsymbol{b}, b_{n+1}) = \{(\boldsymbol{x} + \boldsymbol{b},\ x_{n+1} + b_{n+1}) \mid (\boldsymbol{x}, x_{n+1}) \in Cone\}.$$

Let $f : \mathbf{R}^n \to \mathbf{R}^1$ satisfy

$$(Cone + (\boldsymbol{x}, f(\boldsymbol{x}))) \cap G = \emptyset \quad \text{for any } \boldsymbol{x} \in \mathbf{R}^n, \tag{10}$$

where $G = \{(\boldsymbol{x}, f(\boldsymbol{x})) \mid \boldsymbol{x} \in \mathbf{R}^n\}$ is the graph of f. Prove that f is Lipschitz, and moreover that

$$|f(\boldsymbol{x}) - f(\hat{\boldsymbol{x}})| \leq M|\boldsymbol{x} - \hat{\boldsymbol{x}}| \quad \text{for all } \boldsymbol{x}, \hat{\boldsymbol{x}} \in \mathbf{R}^n. \tag{11}$$

Solution. We note that, directly from the definiitons, the condition $(\boldsymbol{x}, x_{n+1}) \notin Cone + (\boldsymbol{b}, b_{n+1})$ means that

$$x_{n+1} - b_{n+1} \leq M|\boldsymbol{x} - \boldsymbol{b}|.$$

Now let $\boldsymbol{x}, \hat{\boldsymbol{x}} \in \mathbf{R}^n$. From (10), it follows that $(\hat{\boldsymbol{x}}, f(\hat{\boldsymbol{x}})) \notin Cone + (\boldsymbol{x}, f(\boldsymbol{x}))$, which means

$$f(\hat{\boldsymbol{x}}) - f(\boldsymbol{x}) \leq M|\hat{\boldsymbol{x}} - \boldsymbol{x}|.$$

Similarly, $(\boldsymbol{x}, f(\boldsymbol{x})) \notin Cone + (\hat{\boldsymbol{x}}, f(\hat{\boldsymbol{x}}))$, which means

$$f(\boldsymbol{x}) - f(\hat{\boldsymbol{x}}) \leq M|\hat{\boldsymbol{x}} - \boldsymbol{x}|.$$

Combining these inequalities, we obtain (11).