Doubly Stochastic Graph Matrices

Xiao-Dong ZHANG (张晓东)
Shanghai Jiao Tong University
xiaodong@sjtu.edu.cn

2012 Shanghai Conference on Algebraic Combinatorics
August 17th to 22th, 2012
Outline

- Introduction to the doubly stochastic graph matrix $\Omega(G)$.
- Two Merris’ Conjectures: The smallest element of $\Omega(G)$ and the algebraic connectivity.
- Two Merris’ Questions: The element of $\Omega(G)$ and the structure of its graph.
Outline

- Introduction to the doubly stochastic graph matrix $\Omega(G)$.
- Two Merris’ Conjectures: The smallest element of $\Omega(G)$ and the algebraic connectivity.
- Two Merris’ Questions: The element of $\Omega(G)$ and the structure of its graph.
Outline

- Introduction to the doubly stochastic graph matrix $\Omega(G)$.
- Two Merris’ Conjectures: The smallest element of $\Omega(G)$ and the algebraic connectivity.
- Two Merris’ Questions: The element of $\Omega(G)$ and the structure of its graph.
Definition and Background

- Let $G = (V, E)$ be a simple graph (without loop and multiedges) with vertex set $V$ and edge set $E(G)$.
- $A(G) = (a_{ij})$ : the adjacency matrices of $G$ with $a_{ij} = 1$ if $v_i \sim v_j$; $a_{ij} = 0$ otherwise.
- $D(G) = diag(d_1, \cdots, d_n)$ degree diagonal matrix with $d_i$ being the degree of vertex $v_i$. 
Definition and Background

- Let $G = (V, E)$ be a simple graph (without loop and multiedges) with vertex set $V$ and edge set $E(G)$.
- $A(G) = (a_{ij})$ : the adjacency matrices of $G$ with $a_{ij} = 1$ if $v_i \sim v_j$; $a_{ij} = 0$ otherwise.
- $D(G) = \text{diag}(d_1, \cdots, d_n)$ degree diagonal matrix with $d_i$ being the degree of vertex $v_i$. 
Definition and Background

- Let $G = (V, E)$ be a simple graph (without loop and multiedges) with vertex set $V$ and edge set $E(G)$.
- $A(G) = (a_{ij})$: the adjacency matrices of $G$ with $a_{ij} = 1$ if $v_i \sim v_j$; $a_{ij} = 0$ otherwise.
- $D(G) = diag(d_1, \cdots, d_n)$ degree diagonal matrix with $d_i$ being the degree of vertex $v_i$. 
The Laplacian matrix of $G$: $L(G) = D(G) - A(G)$.

$L(G)$ is positive semi-definite and singular.

The eigenvalues of $L(G)$ are denoted by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0;$$

called the Laplacian eigenvalues of $G$. In particular, $\lambda_{n-1}(G)$ is called the algebraic connectivity of $G$ and denoted by $\alpha(G)$.

For example, let $G$ be a graph of order 5 with five edges as follows:
The Laplacian matrix of $G$: $L(G) = D(G) - A(G)$.

$L(G)$ is positive semi-definite and singular.

The eigenvalues of $L(G)$ are denoted by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0;$$

called the Laplacian eigenvalues of $G$. In particular, $\lambda_{n-1}(G)$ is called the algebraic connectivity of $G$ and denoted by $\alpha(G)$.

For example, let $G$ be a graph of order 5 with five edges as follows:
The Laplacian matrix of $G$: $L(G) = D(G) - A(G)$.

$L(G)$ is positive semi-definite and singular.

The eigenvalues of $L(G)$ are denoted by

$$
\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0;
$$

called the Laplacian eigenvalues of $G$. In particular, $\lambda_{n-1}(G)$ is called the algebraic connectivity of $G$ and denoted by $\alpha(G)$.

For example, let $G$ be a graph of order 5 with five edges as follows:
The **Laplacian matrix** of $G$: $L(G) = D(G) - A(G)$.

$L(G)$ is positive semi-definite and singular.

The eigenvalues of $L(G)$ are denoted by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0;$$

called the **Laplacian eigenvalues** of $G$. In particular, $\lambda_{n-1}(G)$ is called the algebraic connectivity of $G$ and denoted by $\alpha(G)$.

For example, let $G$ be a graph of order 5 with five edges as follows:
\[ A(G) = \begin{pmatrix}
0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}. \]
$L(G) = \begin{pmatrix}
3 & -1 & 0 & -1 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
-1 & 0 & -1 & 2 & 0 \\
-1 & 0 & 0 & 0 & 1
\end{pmatrix}$.

- The Laplacian eigenvalue of $G$ is

$$4.4812 > 2.6889 > 2 > 0.8299 > 0.0000$$

- $\alpha(G) = 0.8299$
The Laplacian eigenvalue of $G$ is

$$4.4812 > 2.6889 > 2 > 0.8299 > 0.0000$$

$\alpha(G) = 0.8299$
• (Fiedler 1973) \( G \) is connected if and only if \( \alpha(G) > 0 \).

• (Fiedler 1973) \( \alpha(G) \leq \nu(G) \leq \eta(G) \), where \( \nu(G) \) and \( \eta(G) \) are node connectivity and edge connectivity of graphs.

• \( \alpha(G) \) can be served as a measure of connectivity.

• Matrix-tree Theorem: Let \( G \) be a graph with the Laplacian matrix \( L(G) \). Denoted by \( L(G)(i|j) \) be submatrix of \( L(G) \) by deleting the \( i \)-th row and \( j \)-th column. Then the number \( \tau(G) \) of the spanning trees of \( G \) is equal to

\[
\tau(G) = (-1)^{i+j} \det L(G)(i|j) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.
\]
(Fiedler 1973) $G$ is connected if and only if $\alpha(G) > 0$.

(Fiedler 1973) $\alpha(G) \leq \nu(G) \leq \eta(G)$, where $\nu(G)$ and $\eta(G)$ are node connectivity and edge connectivity of graphs.

$\alpha(G)$ can be served as a measure of connectivity.

Matrix-tree Theorem: Let $G$ be a graph with the Laplacian matrix $L(G)$. Denoted by $L(G)(i|j)$ be submatrix of $L(G)$ by deleting the $i$–th row and $j$–th column. Then the number $\tau(G)$ of the spanning trees of $G$ is equal to

$$\tau(G) = (-1)^{i+j} \det L(G)(i|j) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$
• (Fiedler 1973) $G$ is connected if and only if $\alpha(G) > 0$.
• (Fiedler 1973) $\alpha(G) \leq \nu(G) \leq \eta(G)$, where $\nu(G)$ and $\eta(G)$ are node connectivity and edge connectivity of graphs.
• $\alpha(G)$ can be served as a measure of connectivity.

Matrix-tree Theorem: Let $G$ be a graph with the Laplacian matrix $L(G)$. Denoted by $L(G)(i|j)$ be submatrix of $L(G)$ by deleting the $i$—th row and $j$—th column. Then the number $\tau(G)$ of the spanning trees of $G$ is equal to

$$\tau(G) = (-1)^{i+j} \det L(G)(i|j) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$
(Fiedler 1973) $G$ is connected if and only if $\alpha(G) > 0$.

(Fiedler 1973) $\alpha(G) \leq \nu(G) \leq \eta(G)$, where $\nu(G)$ and $\eta(G)$ are node connectivity and edge connectivity of graphs.

$\alpha(G)$ can be served as a measure of connectivity.

Matrix-tree Theorem: Let $G$ be a graph with the Laplacian matrix $L(G)$. Denoted by $L(G)(i|j)$ be submatrix of $L(G)$ by deleting the $i$–th row and $j$–th column. Then the number $\tau(G)$ of the spanning trees of $G$ is equal to 

$$\tau(G) = (-1)^{i+j} \det L(G)(i|j) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$
Definition:

$$\Omega(G) = (I_n + L(G))^{-1}$$

is a doubly stochastic matrix and called the *doubly stochastic graph matrix* of $G$.

- $G$ is connected if and only if $\Omega(G)$ is entrywise positive.
- It was introduced in "V.E. Golender, V.V. Drboglav, A.B. Rosenblit, Graph potentials method and its application for chemical information processing", Journal of chemical information and computer sciences, 21(1981) 126-204."
Definition:

\[ \Omega(G) = (I_n + L(G))^{-1} \]

is a doubly stochastic matrix and called the *doubly stochastic graph matrix* of \( G \).

\( G \) is connected if and only if \( \Omega(G) \) is entrywise positive.

It was introduced in "V.E. Golender, V.V. Drboglav, A.B. Rosenblit, Graph potentials method and its application for chemical information processing", Journal of chemical information and computer sciences, 21(1981) 126-204."
Definition:

\[ \Omega(G) = (I_n + L(G))^{-1} \]

is a doubly stochastic matrix and called the *doubly stochastic graph matrix* of \( G \).

\( G \) is connected if and only if \( \Omega(G) \) is entrywise positive.

It was introduced in "V.E.Golender, V.V. Drboglav, A.B. Rosenblit, Graph potentials method and its application for chemical information processing", Journal of chemical information and computer sciences, 21(1981) 126-204."
This matrix is also introduced and called doubly stochastic graph matrix in Merris, Doubly stochastic graph matrices, Univ. Beograd. Pub. ELektrotehn. Fak, 8(1997) 64-71; and Doubly stochastic graph matrices, (II), Linear Algebra and Multilinear Algebra, 45(1998) 275-285.

The matrix was also introduced in P.Yu. Chebotarev and E.V.Shamis, The matrix-forest theorem and measuring relations in small social groups, Automation and Remote Control, 58(1997) 1505-1514.
This matrix is also introduced and called doubly stochastic graph matrix in Merris, Doubly stochastic graph matrices, Univ. Beograd. Pub. ELektrotehn. Fak, 8(1997) 64-71; and Doubly stochastic graph matrices, (II), Linear Algebra and Multilinear Algebra, 45(1998) 275-285.

The matrix was also introduced in P.Yu. Chebotarev and E.V. Shamis, The matrix-forest theorem and measuring relations in small social groups, Automation and Remote Control, 58(1997) 1505-1514.
Given a graph, how should one evaluate the proximity between its vertices?

Proximity measures for the vertices of graphs arise in many applied settings, such as chemistry, urban planning, political science, epidemiology, crystallography, sociology.

The standard distance function is the length of the shortest path. But is it not worth taking into account the number of paths between vertices?
Given a graph, how should one evaluate the proximity between its vertices?

Proximity measures for the vertices of graphs arise in many applied settings, such as chemistry, urban planning, political science, epidemiology, crystallography, sociology.

The standard distance function is the length of the shortest path. But is it not worth taking into account the number of paths between vertices?
Given a graph, how should one evaluate the proximity between its vertices?

Proximity measures for the vertices of graphs arise in many applied settings, such as chemistry, urban planning, political science, epidemiology, crystallography, sociology

The standard distance function is the length of the shortest path. But is it not worth taking into account the number of paths between vertices?
The doubly stochastic graph matrix is also called "relative forest accessibility" or "accessibilities"

The entries $\omega_{ij}$ of $\Omega(G)$ can be used to measure the proximity between vertices.

A distinctive feature of the index of proximity is its normalization: the sum of the accessibilities of all vertices from a given vertex and the sum of the accessibilities of a given vertex from all vertices of a graph are equal to unity.

Each $i$th row of the matrix $\Omega(G)$ can be treated as a probability distribution somehow related to the vertex $v_i$. 
The doubly stochastic graph matrix is also called ”relative forest accessibility ” or ”accessibilities”

The entries $\omega_{ij}$ of $\Omega(G)$ can be used to measure the proximity between vertices.

A distinctive feature of the index of proximity is its normalization: the sum of the accessibilities of all vertices from a given vertex and the sum of the accessibilities of a given vertex from all vertices of a graph are equal to unity.

Each $i$th row of the matrix $\Omega(G)$ can be treated as a probability distribution somehow related to the vertex $v_i$. 
The doubly stochastic graph matrix is also called "relative forest accessibility" or "accessibilities".

The entries $\omega_{ij}$ of $\Omega(G)$ can be used to measure the proximity between vertices.

A distinctive feature of the index of proximity is its normalization: the sum of the accessibilities of all vertices from a given vertex and the sum of the accessibilities of a given vertex from all vertices of a graph are equal to unity.

Each $i$th row of the matrix $\Omega(G)$ can be treated as a probability distribution somehow related to the vertex $v_i$. 
• The doubly stochastic graph matrix is also called "relative forest accessibility" or "accessibilities"

• The entries $\omega_{ij}$ of $\Omega(G)$ can be used to measure the proximity between vertices.

• A distinctive feature of the index of proximity is its normalization: the sum of the accessibilities of all vertices from a given vertex and the sum of the accessibilities of a given vertex from all vertices of a graph are equal to unity.

• each $i$th row of the matrix $\Omega(G)$ can be treated as a probability distribution somehow related to the vertex $v_i$. 
\[
\Omega(G) = \begin{pmatrix}
0.3784 & 0.1622 & 0.1081 & 0.1622 & 0.1892 \\
0.1622 & 0.4505 & 0.1892 & 0.1171 & 0.0811 \\
0.1081 & 0.1892 & 0.4595 & 0.1892 & 0.0541 \\
0.1622 & 0.1171 & 0.1892 & 0.4505 & 0.0811 \\
0.1892 & 0.0811 & 0.0541 & 0.0811 & 0.5946
\end{pmatrix}.
\]

- \(\omega_{35}\) is the smallest element in \(\Omega(G)\), while the distance between vertices \(v_3\) and \(v_5\) is the largest in \(G\). There are some correlation relations.

- \(\omega_{55}\) and \(\omega_{11}\) are the largest and the smallest element on the main diagonal, respectively. They correspond to the smallest and largest degrees, respectively.

\[
\alpha(G) = 0.8299 > 2(5 + 1) \times 0.0541.
\]
Some known results

- A graph theoretical interpretation for the entries of $\Omega(G)$.
- Fix a graph $G$, $\mathcal{F}$: the set of all spanning forests $F$ of $G$.
  $\gamma(F)$: the product of the numbers of vertices in the connected components of $F$.
  $\gamma_i(F)$: the product of numbers of vertices in the connected components of $F$ that does not contain vertex $v_i$.
  $\mathcal{F}_{i,j} = \{F \in \mathcal{F} : v_i \text{ and } v_j \text{ belong to the same component of } F\}$.
Some known results

- A graph theoretical interpretation for the entries of $\Omega(G)$.
- Fix a graph $G$, $\mathcal{F}$: the set of all spanning forests $F$ of $G$.
  $\gamma(F)$: the product of the numbers of vertices in the connected components of $F$.
  $\gamma_i(F)$: the product of numbers of vertices in the connected components of $F$ that does not contain vertex $v_i$.
  $\mathcal{F}_{i,j} = \{F \in \mathcal{F} : v_i$ and $v_j$ belong to the same component of $F\}$. 
Theorem 1

(Merris 1997, Golender et.al. 1981) Let $G$ be a graph of order $n \geq 2$. Then the $(i, j)$ entry of $\Omega(G)$ is equal to

$$\omega_{ij} = \frac{\sum_{F \in F(i,j)} \gamma_i(F)}{\sum_{F \in F} \gamma(F)}$$
Theorem 2

(Merris 1998) Let $G$ be a graph of order $n \geq 2$ and $\Omega(G) = (\omega_{ij})$ with $\omega(G) = \min\{\omega_{ij} : 1 \leq i, j \leq n\}$. Then

1. $G$ is connected if and only if $\omega(G) > 0$.
2. $\alpha(G) \geq (1 + \alpha(G))n\omega(G)$.
3. $\omega(G) \leq \frac{1}{n+1}$ with equality if and only if $G = K_n$. 
Some Remarks:

- From (2) in Theorem 2, we can get $\omega(G) < \frac{1}{n}$, which can be improved. Hence (2) may be improved, at least when $\alpha(G)$ is more than $\frac{n+2}{n}$.

- Theorem 2 may suggest that the possibility of viewing $\omega(G)$ as a quantitative measure of connectivity. Most graph theorists would probably agree that adding an edge to a graph cannot make it less connected. $\omega(G)$ fails this test, i.e., there exists a graph $G$ such that

$$\omega(G) > \omega(G + e).$$
Some Remarks:

From (2) in Theorem 2, we can get $\omega(G) < \frac{1}{n}$, which can be improved. Hence (2) may be improved, at least when $\alpha(G)$ is more than $\frac{n+2}{n}$.

Theorem 2 may suggest that the possibility of viewing $\omega(G)$ as a quantitative measure of connectivity. Most graph theorists would probably agree that adding an edge to a graph cannot make it less connected. $\omega(G)$ fails this test, i.e., there exists a graph $G$ such that

$$\omega(G) > \omega(G + e).$$
Some Remarks:

From (2) in Theorem 2, we can get $\omega(G) < \frac{1}{n}$, which can be improved. Hence (2) may be improved, at least when $\alpha(G)$ is more than $\frac{n+2}{n}$.

Theorem 2 may suggest that the possibility of viewing $\omega(G)$ as a quantitative measure of connectivity. Most graph theorists would probably agree that adding an edge to a graph cannot make it less connected. $\omega(G)$ fails this test, i.e., there exists a graph $G$ such that

$$\omega(G) > \omega(G + e).$$
Two Merris’ Conjectures

Conjecture 3

(Merris 1998) Let $G$ be a graph of order $n \geq 2$. Then

$$\alpha(G) \geq 2(n + 1)\omega(G).$$

Conjecture 4

(Merris 1998) Let $E_n$ be the degree antiregular graph of order $n \geq 3$; i.e., the unique connected graph whose vertex degrees attain all values between 1 and $n - 1$. Then

$$\omega(E_n) = \frac{1}{2(n + 1)}.$$
On Conjecture 4, This assertion has been proved to be true.

**Theorem 5**

*(Berman, Z. 2000) Let $E_n$ be the degree antiregular graph of order $n \geq 3$; i.e., the unique connected graph whose vertex degrees attain all values between 1 and $n - 1$. Then

$$\omega(E_n) = \frac{1}{2(n + 1)}.$$*
On Conjecture 3. There are the following results and example.

**Theorem 6**

(Z, Wu 2005) Let $T$ be a tree of order $n \geq 2$. Then

$$\frac{\sqrt{5}}{(\frac{3+\sqrt{5}}{2})^n - (\frac{3-\sqrt{5}}{2})^n} \leq \omega(T) \leq \frac{1}{2(n+1)}$$

with right equality if and only if $T$ is a star and left equality if and only if $T$ is a path.

Can we insert the term $\frac{\alpha(T)}{2(n+1)}$ in Inequality (1)? I.e.,

$$\omega(T) \leq \frac{\alpha(T)}{2(n+1)} \leq \frac{1}{2(n+1)}.$$

Clearly, Conjecture 3 is true for the star $K_{1,n-1}$, since

$$\alpha(K_{1,n-1}) = 1 = 2(n+1)\omega(K_{1,n-1})$$
Example Let $T$ be a tree of order 7 as in Figure 1

FIGURE 1
Then the eigenvalues of $L(T)$ are 5.2618, 3.3399, 1, 1, 1, 0.3983, 0. Moreover, $\Omega(T) =$

\[
\begin{pmatrix}
0.3158 & 0.1053 & 0.1579 & 0.1579 & 0.1579 & 0.0526 & 0.0526 \\
0.1053 & 0.3684 & 0.0526 & 0.0526 & 0.0526 & 0.1842 & 0.1842 \\
0.1579 & 0.0526 & 0.5789 & 0.0789 & 0.0789 & 0.0263 & 0.0263 \\
0.1579 & 0.0526 & 0.0789 & 0.5789 & 0.0789 & 0.0263 & 0.0263 \\
0.1579 & 0.0526 & 0.0789 & 0.0789 & 0.5789 & 0.0263 & 0.0263 \\
0.0526 & 0.1842 & 0.0263 & 0.0263 & 0.0263 & 0.5981 & 0.0921 \\
0.0526 & 0.1842 & 0.0263 & 0.0263 & 0.0263 & 0.0921 & 0.5981 \\
\end{pmatrix}
\]

Hence $\alpha(T) = 0.3983 < 2 \times (7 + 1) \times 0.0263 = 2(n + 1)\omega(T)$.

Hence, in generally, Conjecture 3 is not correct.
$T_{r,s}$: the *doubly star tree* of order $r + s + 2 = n$, which is obtained by joining the dominant vertex of a star $K_{1,r}$ and the dominant vertex of a star $K_{1,s}$ with an edge.

**Theorem 7**

(Z 2009) Let $T$ be a doubly star tree $T_{s,n-s-2}$ of order $n$ with diameter 3. If $T_{s,n-s-2}$ is one of $T_{2,2}, T_{1,8}, T_{1,7}, \cdots , T_{1,1}$, then

$$\alpha(T_{s,n-s-2}) \geq 2(n + 1)\omega(T_{s,n-s-2}).$$

If $T_{s,n-s-2}$ is not any one of $T_{2,2}, T_{1,8}, T_{1,7}, \cdots , T_{1,1}$, then

$$\alpha(T_{s,n-s-2}) < 2(n + 1)\omega(T_{s,n-s-2}).$$
(Z 2009) Let $T$ be a tree of order $n \geq 4$ with diameter $d$. If
$$d \geq \frac{\lg 3 + 3 \lg n}{\lg (3 + \sqrt{5}) - 1} - 1,$$
then $\alpha(T) \geq 2(n + 1)\omega(T)$.

Theorem 9

(Z 2009) Let $T$ be a tree of order $n$ with $p$ non-pendant vertices. Then
$$\alpha(T) \geq \frac{(n + p)\omega(T)}{1 - (n + p)\omega(T)}$$
with equality if and only if $T$ is the star graph $K_{1,n-1}$. 


The smallest element and Diameter

**Theorem 10**

(Z 2009) Let $T$ be a tree of order $n$ with diameter $d$. Then

$$\omega(T) \leq \frac{\sqrt{5}}{\left(\frac{3+\sqrt{5}}{2}\right)^{d+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{d+1}}$$

with equality if and only if $T$ is a path of order $n$.

**Theorem 11**

(Z 2009) Let $T$ be a tree of order $n$. If $\omega(T) = \omega_{k,l}$, then

1. $v_k$ and $v_l$ are two pendant vertices, i.e., the degrees of vertices $v_k$ and $v_l$ are 1.
2. If diameter $d$ of $T$ is no more than 4, then the distance between $v_k$ and $v_l$ is equal to $d$. 

Merris’ Question

Theorem 12

(Merris 1997) Let $G$ be a simple graph of order $n \geq 2$. If $d(v_i) < n - 1$, then there exists a $j$ such that $\omega_{ij} < \frac{1}{n+1}$.

- It seems natural to wonder whether what governs whether $\omega_{ij}$ is more or less than $\frac{1}{n+1}$.
- Does $(v_i, v_j) \in E(G)$ imply $\omega_{ij} \geq \frac{1}{n+1}$? The answer is NO.
- What about the other way around? Does $\omega_{ij} \geq \frac{1}{n+1}$ imply $(v_i, v_j) \in E(G)$? The answer is NO.
Merris’ Question

Theorem 12

(Merris 1997) Let $G$ be a simple graph of order $n \geq 2$. If $d(v_i) < n - 1$, then there exists a $j$ such that $\omega_{ij} < \frac{1}{n+1}$.

- It seems natural to wonder whether what governs whether $\omega_{ij}$ is more or less than $\frac{1}{n+1}$.
- Does $(v_i, v_j) \in E(G)$ imply $\omega_{ij} \geq \frac{1}{n+1}$? The answer is NO.
- What about the other way around? Does $\omega_{ij} \geq \frac{1}{n+1}$ imply $(v_i, v_j) \in E(G)$? The answer is NO.
Merris’ Question

Theorem 12

\[(\text{Merris 1997})\] Let \( G \) be a simple graph of order \( n \geq 2 \). If 
\[d(v_i) < n - 1, \text{ then there exists a } j \text{ such that } \omega_{ij} < \frac{1}{n+1}.\]

- It seems natural to wonder whether what governs whether \( \omega_{ij} \) is more or less than \( \frac{1}{n+1} \).
- Does \((v_i, v_j) \in E(G)\) imply \( \omega_{ij} \geq \frac{1}{n+1}\)? The answer is NO.
- What about the other way around? Does \( \omega_{ij} \geq \frac{1}{n+1}\) imply \((v_i, v_j) \in E(G)\)? The answer is NO.
Theorem 13

\textit{(Merris 1998) Let }G\textit{ be a simple graph of order } n \geq 2 \textit{. If } (v_i, v_j) \in E(G), \textit{ then } \omega_{ij} \geq \frac{4}{n^2+4n}.

- \textbf{Merris’s Question 1} Does there exist a constant } c, \textit{ independent of } n, \textit{ such that } \omega_{ij} \geq \frac{c}{n} \textit{ whenever } (v_i, v_j) \in E(G)\text{?}

- The answer is \textbf{NO}.
Theorem 13

(Merris 1998) Let $G$ be a simple graph of order $n \geq 2$. If $(v_i, v_j) \in E(G)$, then $\omega_{ij} \geq \frac{4}{n^2+4n}$.

- **Merris’s Question 1** Does there exist a constant $c$, independent of $n$, such that $\omega_{ij} \geq \frac{c}{n}$ whenever $(v_i, v_j) \in E(G)$?
- The answer is NO.
Theorem 14

(Z 2005) Let $T$ be a tree of order $n \geq 2$ with the doubly stochastic matrix $\Omega(T) = (\omega_{ij})$. If $(v_i, v_j) \in E(T)$, then

$$\omega_{ij} \geq \frac{4}{(\lfloor \frac{n}{2} \rfloor + 3)(\lceil \frac{n}{2} \rceil + 3) - 4}$$

with equality if and only if $T$ is a doubly star tree $T_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil - 1}$, where $\lfloor x \rfloor$ ($\lceil x \rceil$) is the largest (smallest) integer no more (less) than $x$. 
Remark Let $T_{\lfloor \frac{n}{2} \rfloor-1, \lceil \frac{n}{2} \rceil-1}$ be a doubly star tree and let
\[ \{v_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor+1}\} \]
be an edge of $T_{\lfloor \frac{n}{2} \rfloor-1, \lceil \frac{n}{2} \rceil-1}$ joining the center of
$K_{1, \lfloor \frac{n}{2} \rfloor-1}$ and the center of $K_{1, \lceil \frac{n}{2} \rceil-1}$. Then
\[
\omega_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor+1} = \frac{4}{(\lceil \frac{n}{2} \rceil + 3)(\lfloor \frac{n}{2} \rfloor + 3) - 4} \approx \frac{16}{n^2}.
\]
Hence there does not exist a constant $c$, independent of $n$ such that
\[
\omega_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor+1} \approx \frac{16}{n^2} \geq \frac{c}{n}.
\]
We answer Merris’ question 1.
Theorem 15

***(Merris 1998)*** Let $G$ be any graph of order $n \geq 2$. Then $\omega_{ii} > 2\omega_{ij}$ for any $j \neq i$.

- This Theorem has a natural interpretation, namely, each vertex is more "accessible" from itself than from any other vertex.

Theorem 16

**((Chebotarev, Shamis, 1997)*** Let $G$ be a graph of order $n \geq 2$. Then

$$\omega_{ij} + \omega_{ik} - \omega_{jk} \leq \omega_{ii}.$$ 

This inequality is called triangle inequality for proximities.
Proximity of Vertex

**Theorem 15**

(Merris 1998) Let $G$ be any graph of order $n \geq 2$. Then $\omega_{ii} > 2\omega_{ij}$ for any $j \neq i$.

This Theorem has a natural interpretation, namely, each vertex is more "accessible" from itself than from any other vertex.

**Theorem 16**

((Chebotarev, Shamis, 1997) Let $G$ be a graph of order $n \geq 2$. Then

\[ \omega_{ij} + \omega_{ik} - \omega_{jk} \leq \omega_{ii}. \]

This inequality is called triangle inequality for proximities.
Theorem 17

(Chebotarev, Shamis 1997) Let \( \rho(v_i, v_j) = \omega_{ii} + \omega_{jj} - 2\omega_{ij} \). Then is a distance function with the axioms of metric.

\[
\begin{align*}
  r(i) &= \sum_{j=1, j \neq i}^{n} \rho(v_i, v_j). \\
  s(G) &= \min\{r(i) : 1 \leq i \leq n\}, \\
  S(G) &= \max\{r(i) : 1 \leq i \leq n\}
\end{align*}
\]

- \( r(k) = S(G) \) means that \( v_k \) is a most remote vertex.
- \( r(k) = s(G) \) means that \( v_k \) is a least remote vertex.
- \( \omega_{ii} \) is a measure of the "solitariness" of vertex \( v_i \).
Theorem 17

(Chebotarev, Shamis 1997) Let \( \rho(v_i, v_j) = \omega_{ii} + \omega_{jj} - 2\omega_{ij} \). Then it is a distance function with the axioms of metric.

\[
\begin{align*}
    r(i) &= \sum_{j=1, j \neq i}^{n} \rho(v_i, v_j). \\
    s(G) &= \min\{r(i) : 1 \leq i \leq n\}, \\
    S(G) &= \max\{r(i) : 1 \leq i \leq n\}
\end{align*}
\]

- \( r(k) = S(G) \) means that \( v_k \) is a most remote vertex.
- \( r(k) = s(G) \) means that \( v_k \) is a least remote vertex.
- \( \omega_{ii} \) is a measure of the "solitariness" of vertex \( v_i \).
Theorem 17

(Chebotarev, Shamis 1997) Let $\rho(v_i, v_j) = \omega_{ii} + \omega_{jj} - 2\omega_{ij}$. Then is a distance function with the axioms of metric.

$$r(i) = \sum_{j=1, j \neq i}^{n} \rho(v_i, v_j).$$

$$s(G) = \min\{r(i) : 1 \leq i \leq n\},$$

$$S(G) = \max\{r(i) : 1 \leq i \leq n\}$$

- $r(k) = S(G')$ means that $v_k$ is a most remote vertex
- $r(k) = s(G)$ means that $v_k$ is a least remote vertex.
- $\omega_{ii}$ is a measure of the "solitariness" of vertex $v_i$. 
Merris’ question 2

Theorem 18

(Merris 1998) Let $G$ be a simple graph of order $n \geq 2$. Then $v_k$ is a most remote vertex if and only if $\omega_{kk}$ is a maximal main diagonal entry; and $v_k$ is a least remote vertex if and only if $\omega_{kk}$ is a minimal main diagonal entry.

Merris’ Question 2 (Merris 1998) Does $d(v_k) > d(v_i)$ for all $i \neq k$, imply $r(k) = s(G)$. 
Answer for Merris’ question

On Merris’ Question 2, there are the following results

**Theorem 19**

(Z 2011) Let $G$ be a simple graph. If $d(v_k) > 2d(v_i) - 1$ for all $i \neq k$, then $r(k) = s(G)$.

**Theorem 20**

(Z 2011) There exists a class of graphs with $d(v_k) > d(v_i)$ for all $i \neq k$ and $r(k) > s(G)$. Moreover, the graph has a vertex $v_j$ such that $d(v_k) \leq 2d(v_j) - 3$ for some $j \neq k$. 
Question

Conjecture 21

Let $G$ be a simple connected graph on $n$ vertices $\{v_1, \ldots, v_n\}$ with the doubly stochastic graph matrix $\Omega(G) = (\omega_{ij})$. If $d_k \geq 2d_i - 2$ for all $i = 1, \ldots, k - 1, k + 1, \ldots, n$, does $r(k) = s(G)$ hold?
Reference

P. Yu Chebotarev and E. V. Shamis,
The matrix-forest theorem and measuring relations in small social groups, *Automation and Remote Control*, 58(9)(1997), 1505-1514.

P. Yu Chebotarev and E. V. Shamis,

R. Merris,

R. Merris,
A.Berman, Z

Z

Z, J.X. Wu
Thank you very much for attention!