Decomposition of 3-connected Matroids

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In this talk, we use matroid to represent both finite and infinite matroid.
Sources of Matroids

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Usually, there are “three” ways doing matroid theory: the graph-theoretic way, the geometric way, and the algorithmic way.
Definition of Finite Matroids

Let $E$ be some finite set, $\mathcal{B}$ a set of subsets of $E$. Say $M = (E, \mathcal{B})$ a matroid if the following conditions are satisfied:

(B1) $\mathcal{B}$ is non-empty.

(B2) Whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there is an element $y \in B_2 \setminus B_1$ such that $B_1 - x + y \in \mathcal{B}$. 

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$E$ is the ground set of $M$, and the members of $\mathcal{B}$ are bases of $M$. 
Examples

Example 1. Let $\mathbb{F}$ be a finite matroid and $V(r, \mathbb{F})$ a $r$-dimensional vector space over $\mathbb{F}$. And let $E$ denote the set of all elements of $V(r, \mathbb{F})$, and $B$ the set of all bases of $V(r, \mathbb{F})$. Then $M = (E, B)$ is a matroid and every element in $B$ is a basis of $M$.

Example 2. Let $E$ denote the edge set of a finite graph $G$ and $B$ the set of spanning trees of $G$. Then $M = (E, B)$ is a matroid and every spanning tree of $G$ is a basis of $M$. 
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**Example 2.** Let $E$ denote the edge set of a finite graph $G$ and $\mathcal{B}$ the set of spanning trees of $G$. Then $M = (E, \mathcal{B})$ is a matroid and every spanning tree of $G$ is a basis of $M$. 
Assume $X \subseteq E$. Let

$$r(X) = \max\{|I| : I \subseteq X \text{ is a subset of some basis of } M.\}$$

Then say $r(X)$ is the rank of $X$. 

There are also many other different but equivalent ways to define finite matroids, say, from circuits, from rank, from independent sets, or from a closure operator.
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For a 2-connected finite matroid $M$, Cunningham and Edmonds (in *Canada. J. Math* 32: 734-765, 1980) showed that $M$ can be decomposed into a set of 3-connected finite matroids via a canonical operation known as 2-sum; more concretely, there is a labeled tree that gives a precise description of the way that $M$ is built from the 3-connected pieces.
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The counterexamples given to show Kahn’s Conjecture is false for all fields with at least 7 elements (in *JCTB* 67: 325-343, 1996) have many mutually interacting 3-separations. Thus, it is not possible to decompose a 3-connected matroid across 3-separations in a similar way.
Recently, for any 3-connected finite matroid $M$ with at least 9 elements, Oxley, Semple and Whittle (in *JCTB* 92: 257-293, 2004) described a tree decomposition that displays all non-trivial 3-separations of $M$ up to a certain natural equivalence.

We prove that (in *JCTB* 102: 647-670, 2012) via an operation "reducing", every finite 3-connected representable matroid $M$ with at least 9 elements can be decomposed into a set of sequentially 4-connected matroids and three special matroids; more concretely, there is a labeled tree that gives a precise description of the way that $M$ is built from its pieces.

Sequentially 4-connectivity is weaker than 4-connectivity with many good properties 4-connectivity does not have, such as satisfying duality and a corresponding Tutte’s Wheels and Whirls Theorem, which guarantees the feasibility of implement of mathematics induction for sequentially 4-connected matroids.
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Next, I will introduce one of my recent result about decomposition result of infinite matroids.
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The main feature of these definitions is that even on infinite ground sets matroids have bases, circuits and minors while maintain duality at the same time.
Definition of Matroids in Terms of Bases

Let $E$ be some (possibly infinite) set, $\mathcal{B}$ a set of subsets of $E$, and let $[\mathcal{B}]$ be the set of subsets of elements of $\mathcal{B}$. Say $M = (E, \mathcal{B})$ a matroid if the following conditions are satisfied:

(B1) $\mathcal{B}$ is non-empty.

(B2) Whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there is an element $y \in B_2 \setminus B_1$ such that $B_1 - x + y \in \mathcal{B}$.

(BM) Set $\mathcal{I} := [\mathcal{B}]$. Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.
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When $E$ is finite, the definition is equivalent to the former one.
Connectivity of Matroids

Let \((X, Y)\) be a partition of \(E\), \(B_X\) and \(B_Y\) two arbitrary bases of \(M|X\) and \(M|Y\), respectively. Then there is a set \(F \subseteq B_X \cup B_Y\) such that \((B_X \cup B_Y) \setminus F\) is a basis of \(M\). Set \(k = |F|\). (It is known that the number \(k\) does not depend on the choice of \(B_X\) and \(B_Y\).) Say \((X, Y)\) is \((k + 1)\)-separating. And if in addition \(|X|, |Y| \geq k + 1\), then \((X, Y)\) is a \((k + 1)\)-separation.

The matroid \(M\) is \(n\)-connected if it has no \(\ell\)-separation for any \(\ell < n\). For \(M\) finite, these definitions are equivalent to the former one.
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For \(M\) finite, these definitions are equivalent to the former one.
For any connected matroid $M$, Aigner-Horev, Diestel, Postel (in arXiv: 1201.1135v1) proved that there is a unique tree $T$ such that the nodes of $T$ corresponding to minors of $M$ that are either 3-connected or circuits or cocircuits.

Recently, we prove Theorem (in preparation) For any 3-connected matroid $M$ with at least 9 elements, there is a tree decomposition of $M$, which displays all non-trivial 3-separation of $M$ up to a certain natural equivalence.

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In the proof, first we find a special set $G$ and define a labeled graph $T_G$ corresponding to it; and then we show the labeled graph is a tree-decomposition with the satisfied properties.
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The paper is about 40 pages, a little longer than theirs.
Thank You!