Some Counting Problems in Archimedean Tilings

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1 Introduction

2 Main Results

3 What’s next?
1 Introduction

2 Main Results

3 What's next?
Let \( \vec{u} \) and \( \vec{v} \) be two linearly independent real vectors in \( \mathbb{R}^2 \). The set of all points \( P = m\vec{u} + n\vec{v} \) with integral \( m, n \) is called a *general lattice* \( \Lambda \) with basis \( \vec{u} \) and \( \vec{v} \).
A point of the lattice $\Lambda$ is called a lattice point.
Specially, if $\vec{u}$ and $\vec{v}$ are mutually orthogonal unit vectors, the lattice $\Lambda$ is called an *integer lattice* $\mathbb{Z}^2$. 
The number of lattice points in a circle

Let $D(n)$ be a circle centered at a lattice point and with radius $r = n$. In 1837, C.F.Gauss published a result discussing the number $N(n)$ of lattice points lying inside or on the boundary of $D(n)$, where $n \in \mathbb{Z}^+$. Furthermore, C.F.Gauss showed that the ratio $N(n)/n^2$ tends to $\pi$ as $n$ tends to $\infty$. 

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Some Counting Problems in Archimedean Tilings
Minkowski’s Theorem

In 1896, Hermann Minkowski (1864-1909) proved Minkowski’s theorem, and developed a new research area, the Geometry of Numbers.
Minkowski’s Theorem

Any convex set of area greater than or equal to 4 which is symmetric about a lattice point, contains at least one other lattice point in its interior or on its boundary.
Blichfeldt’s Theorem

In 1914, Hans Frederik Blichfeldt published a theorem from which a great portion of the geometry of numbers follows. This theorem mainly discusses the relationship between the area of any bounded set in $\mathbb{R}^2$ and the number of lattice points.
Blichfeldt’s Theorem

Any bounded set $D$ of area $s$ in $\mathbb{R}^2$ can be translated on the integer lattice $\Lambda$ so as to cover at least $[s]+1$ lattice points.
Let $P$ be a lattice polygon of $\mathbb{R}^2$, i.e., the vertices of $P$ are points of the integer lattice $\mathbb{Z}^2$. Let $I(P)$ be the number of lattice points in the interior of $P$ and $B(P)$ the number of lattice points of its boundary. In 1899, George Alexander Pick (1859 - 1942) gave the Pick’s Theorem as follow:

$$\text{Area}(P) = I(P) + B(P)/2 - 1.$$
A plane tiling $\mathcal{T}$ is a countable family of closed sets, that is, $\mathcal{T} = \{T_1, T_2, \cdots\}$, which covers the plane without gaps or overlaps. $T_1, T_2, \cdots$ are called the tiles of $\mathcal{T}$. 
Introduction

A tiling is called monohedral if every tile in the tiling $\mathcal{T}$ is congruent (directly or reflectively) to one fixed set $T$.

An edge-to-edge tiling means that an edge of $\mathcal{T}$ is exactly the full common side of two adjacent tiles.

A vertex around which, in cyclic order, we have a regular $n_1$-gon, $n_2$-gon, etc., is said to be of type $n_1.n_2.\cdots$. 
Let an Archimedean tiling be an edge-to-edge tiling by regular polygons with all vertices being of the same type. An Archimedean tiling with vertex type $n_1.n_2.\cdots.n_k$ is called $(n_1.n_2.\cdots.n_k)$-tiling.

There are 11 Archimedean tilings.
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Pick-type Theorem in (6,6,6)-tiling

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Some Counting Problems in Archimedean Tilings
Counting Problems in the Archimedean (6.6.6)-tiling

Let $H$ be the vertex set of (6.6.6)-tiling. A point of $H$ is called an $H$-point. In fact, the set $H$ can be regarded as a disjoint union of two sets $H^+$ and $H^-$. 
The number of $H$-points in a circle

- $D(n)$: a circle of radius $r = n$ ($n \in \mathbb{Z}^+$) centered at an $H$-point.

- $N_{D(n)}(H)$: the number of $H$-points lying inside or on the boundary of $D(n)$.

We present an algorithm to calculate the number $N_{D(n)}(H)$ and obtain the following table.
The number of $H$-points in a circle

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The number of $H$-points in a circle

**Theorem.** Let $D(n)$ be a circle centered at the origin $O$ and the radius $r = n$, where $n \in \mathbb{Z}^+$. Then

$$\lim_{n \to \infty} \frac{\mathcal{N}_{D(n)}(H)}{n^2} = \frac{4\sqrt{3}\pi}{9}.$$
The number of $H$-points in a circle
A Minkowski-type Theorem for $H$-points

**Theorem.** Let $D$ be a convex set which is symmetric with respect to an $H$-point. If the area of $D$ is greater than or equal to $\frac{4}{3}$, then $D$ contains at least one other $H$-point in its interior or on its boundary.
A Blichfeldt-type Theorem for $H$-points

Theorem. [Cao, Yuan, American Mathematical Monthly, 2011] Let $D$ be a bounded set in $\mathbb{R}^2$ with area $s$. If $0 \leq \{s\} < \frac{1}{3}$, then $D$ can be translated so as to cover at least $2\lfloor s \rfloor + 1$ $H$-points. If $\frac{1}{3} \leq \{s\} < 1$, then $D$ can be translated so as to cover at least $2\lfloor s \rfloor + 2$ $H$-points.
The number of $H$-points in the interior of a convex $H$-polygon

**Definition.** Denoted by $H$-polygon $P$ a simple planar polygon whose vertices are all $H$-points. Let $v_H(P)$, $i_H(P)$ and $b_H(P)$ be the number of $H$-points covered by $P$, lying in the interior of $P$ and located on the boundary of $P$.

**Definition.** Let $K$ be a convex $H$-polygon, denoted by $G(v) = \min\{i_H(K) : v_H(K) = v\}$. 
The number of $H$-points in the interior of a convex $H$-polygon

Clearly, $G(3)=G(4)=G(5)=G(6)=0$.

The number of $H$-points in the interior of a convex $H$-polygon

**Theorem.** $G(8) = 2$.

**Theorem.** $G(9) = 4$. 

\[ \text{Diagram of a } H\text{-polygon with } \text{H-points marked} \]
The number of $H$-points in the interior of a convex $H$-polygon

**Theorem.** $G(10) = 6$.

**Theorem.** $G(v) \geq \left[ \frac{v^3}{16\pi^2} - \frac{v}{4} + \frac{1}{2} \right] - 1$, where $v$ is the number of vertices of the convex $H$-polygon.
Let $C$ be the vertex set of (3.6.3.6)-tiling and let $C$-point be a point of $C$. For convenience, we classify the set $C$ into two sets $C_1$ and $C_2$. A point in $C_1$ is called $C_1$-point, and a point in $C_2$ is called $C_2$-point.
The number of $C$-points in a circle

- $C(\sqrt{n})$: a circle of radius $r = \sqrt{n}$ ($n \in \mathbb{Z}^+$) centered at a $C$-point.
- $N(n)$: the number of $C$-points lying inside or on the boundary of $C(\sqrt{n})$. 

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The number of $C$-points in a circle

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The number of $C$-points in a circle

**Theorem.** Let $D(n)$ be a circle of radius $r = \sqrt{n}$ centered at the origin $O$, where $n \in \mathbb{Z}^+$. Then

$$\lim_{n \to \infty} \frac{N(n)}{n} = \frac{\sqrt{3}}{2}\pi.$$
The number of $C$-points in a circle
A Minkowski-type Theorem for $C'$-points

**Theorem.** Let $D$ be an $M$-set centered at the origin $O$, and $A(D)$ denotes the area of the $M$-set $D$. If $A(D) \geq 4\sqrt{3}$, then $D$ contains $C'$-points other than $O$ either in its interior or on its boundary.
**Theorem.** Let $D$ be a bounded set in $\mathbb{R}^2$ with area $S$. If $0 \leq \left\{ \frac{S}{2\sqrt{3}} \right\} < \frac{1}{4}$, then $D$ can be translated so as to cover at least $3\left\lfloor \frac{S}{2\sqrt{3}} \right\rfloor + 1$ $C$-points; If $\frac{1}{4} \leq \left\{ \frac{S}{2\sqrt{3}} \right\} < \frac{1}{2}$, then $D$ can be translated so as to cover at least $3\left\lfloor \frac{S}{2\sqrt{3}} \right\rfloor + 2$ $C$-points; If $\frac{1}{2} \leq \left\{ \frac{S}{2\sqrt{3}} \right\} < 1$, then $D$ can be translated so as to cover at least $3\left\lfloor \frac{S}{2\sqrt{3}} \right\rfloor + 3$ $C$-points.
Counting Problems in the non-Archimedean 
\((\mathcal{3}^2.\mathcal{6}^2; \mathcal{3}.6.3.6)-\text{tiling}\)

Let \( F \) be the vertex set of \((\mathcal{3}^2.\mathcal{6}^2; \mathcal{3}.6.3.6)-\text{tiling} \) which is not an Archimedean tiling. The set \( F \) can be classified into three sets \( F_0, F_1 \) and \( F_2 \). A point in the set \( F \) and \( F_i (i = 0, 1, 2) \) is called an \( F\text{-point} \) and \( F_i\text{-point} \) respectively.
The number of $F$-points in a circle

- $C(\sqrt{n})$: a circle with its center at an $F$-point and the radius $r = \sqrt{n}$, where $n \in \mathbb{Z}^+$.  
- $N_F(n)$: the number of $F$-points which lie inside or on the boundary of $C(\sqrt{n})$. 

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Some Counting Problems in Archimedean Tilings
The number of $F$-points in a circle

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Table 1: The Number, $N_F(n)$, of $F$-Points for $C(\sqrt{n})$ (centered at an $F_0$-point)
The number of $F$-points in a circle

Table 2: The Number, $N_{F}(n)$, of $F$-Points for $C(\sqrt{n})$ (centered at an $F_1$-point)
The number of $F$-points in a circle

**Theorem.** Let $C(\sqrt{n})$ be a circle with its center at an $F$-point and the radius $r = \sqrt{n}$ where $n \in \mathbb{Z}^+$, and $N_F(n)$ be the number of $F$-points contained inside or on the boundary of $C(\sqrt{n})$. Then

$$\lim_{n \to \infty} \frac{N_F(n)}{n} = \frac{\sqrt{3}\pi}{2}.$$
The number of $F$-points in a circle
**Theorem.** Let $D$ be an $M$-set with area $S_D$, and $O_M$ denote the center of symmetry of $D$.

1. If $O_M$ is an $F_1$-point or $F_2$-point and $S_D \geq 2\sqrt{3}$, then $D$ contains at least one $F$-point in its interior or on its boundary other than $O_M$.

2. If $O_M$ is an $F_0$-point and $S_D \geq 4\sqrt{3}$, then $D$ contains at least two $F$-points in its interior or on its boundary other than $O_M$. 
A Minkowski-type Theorem for $F$-points

The best

It is not difficult to see that the result is best.
**Theorem.** Let $D$ be a bounded set in $\mathbb{R}^2$ with area $S$. If $0 \leq \left\{ \frac{S}{2\sqrt{3}} \right\} < \frac{1}{4}$, then $D$ can be translated so as to cover at least $3\left\lfloor \frac{S}{2\sqrt{3}} \right\rfloor + 1$ $F$-points; If $\frac{1}{4} \leq \left\{ \frac{S}{2\sqrt{3}} \right\} < \frac{1}{2}$, then $D$ can be translated so as to cover at least $3\left\lfloor \frac{S}{2\sqrt{3}} \right\rfloor + 2$ $F$-points; If $\frac{1}{2} \leq \left\{ \frac{S}{2\sqrt{3}} \right\} < 1$, then $D$ can be translated so as to cover at least $3\left\lfloor \frac{S}{2\sqrt{3}} \right\rfloor + 3$ $F$-points.
A Blichfeldt-type Theorem for $F$-points

The best

It is worth to indicate that this theorem is the **best** possible.
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2 Main Results

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The Hamiltonian properties of the sets of vertices of Archimedean tilings

Thank you very much for your attention!