On colorings of mixed hypergraphs

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A mixed hypergraph on a $X$: triple

$$\mathcal{H} = (X, \mathcal{C}, \mathcal{D}),$$

where $\mathcal{C}$ and $\mathcal{D}$ are families of subsets of $X$.

- $\mathcal{C}$-edges, $\mathcal{D}$-edges.
- A set $B \in \mathcal{C} \cap \mathcal{D}$ is called a bi-edge.
- $\mathcal{H}$ is a bi-hypergraph if $\mathcal{C} = \mathcal{D}$, denoted by $(X, \mathcal{B})$.
- $\mathcal{H}$ is $r$-uniform if any edge has $r$ vertices.
A $k$-coloring of $\mathcal{H}$ is a surjective from $X$ into a set of $k$ colors so that each $\mathcal{C}$-edge has two vertices with a Common color and each $\mathcal{D}$-edge has two vertices with Distinct colors. The maximum (minimum) number of colors in any coloring of $\mathcal{H}$ is the upper chromatic number $\overline{\chi}$ (lower chromatic number $\chi$.)
The set of all the values $k$ such that $\mathcal{H}$ has a $k$-coloring is called the **feasible set** of $\mathcal{H}$, denoted by $\Phi(\mathcal{H})$. 
For each $k$, let $r_k$ denote the number of partitions of the vertex set into $k$ color classes. The vector

$$R(\mathcal{H}) = (r_1, r_2, \ldots, r_{\chi})$$

is called the **chromatic spectrum** of $\mathcal{H}$.
Theorem 1 (Jiang, Mubayi, Tuza, Voloshin, West, 2002)
For any finite set $S$ of integers greater than 1, there exists a mixed hypergraph $\mathcal{H}$ such that $\Phi(\mathcal{H}) = S$, 

Theorem 2 (Král, 2004)
Let $S = \{n_1, \ldots, n_t\}$ be a finite set of integers greater than 1. For any positive integer $r_{n_i}$, there exists a mixed hypergraph with chromatic spectrum $(r_{n_1}, \ldots, r_{n_t})$. 
**Theorem 3** (Bujtás and Tuza, 2008)
Let \( r \geq 3 \), \( S \) be a set of positive integers. Then there exists a \( r \)-uniform mixed hypergraph with feasible set \( S \) if and only if

- \( \min(S) \geq r \), or
- \( 2 \leq \min(S) \leq r - 1 \) and \( S \) contains all integers between \( \min(S) \) and \( r - 1 \), or
- \( \min(S) = 1 \) and \( S = \{1, \ldots, \bar{x}\} \) for some natural number \( \bar{x} \geq r - 1 \).
Problems

1. (Bujtás, Tuza, 2008) Determine the chromatic spectrum of $r$-uniform bi-hypergraphs.

2. (Jiang et al 2002) Determine the minimum number of vertices in mixed hypergraphs with given chromatic spectrum.

3. (Bujtás, Tuza, 2008) Determine the minimum number of vertices in $r$-uniform bi-hypergraphs with given feasible set.

4. (Tuza, Voloshin 2008) Determine the minimum numbers of ($C$-, $D$-, bi-) edges in a mixed (bi-) hypergraph with given feasible set $S$. 

Construction:
For any integers $s \geq 2$ and $n_1 \geq \cdots \geq n_s \geq 3$, let
$$X_{n_1,\ldots,n_s} = \{(x_1, \ldots, x_s) | x_j \in [n_j], j \in [s]\}$$
$$\mathcal{B}_{n_1,\ldots,n_s} = \left\{ \left\{(x_1, \ldots, x_s), (y_1, \ldots, y_s), (z_1, \ldots, z_s) \right\} | x_j, y_j, z_j \in [n_j], |\{x_j, y_j, z_j\}| = 2, \forall j \in [s] \right\}.$$
Then $(X_{n_1,\ldots,n_s}, \mathcal{B}_{n_1,\ldots,n_s})$ is a 3-uniform bi-hypergraph, denoted by $\mathcal{H}_{n_1,\ldots,n_s}$. 
3-uniform bi-hypergraph

Theorem 4 (Diao, Zhao, W, 2011, DM)
Let $S = \{n_1, n_2, \ldots, n_t\}$ be a set of positive integers $n_1 > \cdots > n_t \geq 3$ and $t \geq 2$. Then

$$H(n_1, \ldots, n_1, \ldots, n_t, \ldots, n_t)$$

is a 3-uniform bi-hypergraph with the feasible set $S$ and the chromatic spectrum $(r_{n_1}, \ldots, r_{n_t})$. 
A mixed hypergraph $\mathcal{H}$ is a one-realization of $S$ if it is a realization of $S$ and all the entries of the chromatic spectrum of $\mathcal{H}$ are either 0 or 1. Next we shall introduce the development on Problems 2,3,4 for one-realization.
Theorem 5 (Král, 2004, EJC)
Let $S$ be a set of positive integers at least 2. If $\mathcal{H} = (X, C, D)$ is a one-realization of $S$. Then

$$|X| \leq |S| + 2 \max S - \min S.$$
Theorem 6 (Diao, Zhao, W, 2012, EJC)
For integers $s \geq 2$ and $n_1 > n_2 > \cdots > n_s \geq 2$, let $\delta(S)$ denote the number of vertices of the smallest one-realization of $S = \{n_1, n_2, \ldots, n_s\}$. Then

$$\delta(S) = \begin{cases} 2n_1 - n_s, & \text{if } n_1 > n_2 + 1, \\ 2n_1 - n_s - 1, & \text{if } n_1 = n_2 + 1. \end{cases}$$
Theorem 7 (Diao, Zhao, W, 2012, DM)
For integers $s \geq 2$ and $n_1 > n_2 > \cdots > n_s \geq 2$, let $\delta_3(S)$ be the minimum number of vertices of 3-uniform bi-hypergraphs which are one-realizations of $S = \{n_1, n_2, \ldots, n_s\}$. Then

$$\delta_3(S) = \begin{cases} 
6, & \text{if } n_1 = 3, n_2 = 2, \\
2n_1, & \text{if } n_1 > n_2 + 1, \\
2n_1 - 1, & \text{otherwise}.
\end{cases}$$
Theorem 8 (Diao, Zhao, W, 2012)
For integers $s \geq 2$ and $n_1 > n_2 > \cdots > n_s \geq 2$, let $\delta_D(S)$ denote the minimum number of $D$-edges of one-realizations of $S = \{n_1, n_2, \ldots, n_s\}$. Then

$$\delta_D(S) = \begin{cases} 
\frac{n_1(n_1-1)}{2}, & \text{if } n_1 - 1 \notin S, \\
\frac{n_1(n_1-1)}{2} - 1, & \text{if } n_1 - 1 \in S.
\end{cases}$$
Theorem 9 (Diao, Zhao, W, 2012)
For integers $s \geq 2$ and $n_1 > n_2 > \cdots > n_s \geq 2$, let $\delta_C(S)$ be the minimum number of $C$-edges of one-realizations of $S = \{n_1, n_2, \ldots, n_s\}$. Then

$$\delta_C(S) = \begin{cases} 
2n_1 - 2n_s, & \text{if } n_1 - 1, n_s + 1 \notin S, \\
2n_1 - 2n_s - 2, & \text{if } n_1 - 1, n_s + 1 \in S, \\
2n_1 - 2n_s - 1, & \text{otherwise.}
\end{cases}$$


Thank you for listening!