A Poset on the Caterpillar Trees

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Joint Work with Yaokun Wu and Zeying Xu
Kelman (1981) and Csikvári (2010) studied a kind of transformation between trees. $T_1 > T_2$ if $T_1$ can be obtained from $T_2$ by KC transformation.

**Figure:** KC transformation on trees
Csikvári (2010) proved that if $T_1 > T_2$

$$|\text{Hom}(C_m, T_1)| \geq |\text{Hom}(C_m, T_2)|$$

$$|\text{Hom}(P_m, T_1)| \geq |\text{Hom}(P_m, T_2)|$$
A caterpillar is a tree in which all vertices are within distance 1 of a central path. Let $CT_n$ be the set of caterpillar trees with $n$ vertices.
poset of $CT_n$ induced by KC transformation is a subposet of $T_n$. 

Figure: $T_7$
Möbius function

Zeta function: $\zeta(x, y) = 1$ iff $x \leq y$ in $P$.

The inverse of the zeta function in the incidence algebra is called Möbius function.
What We Know About $CT_n$

- The Möbius function of $CT_n$ alternates in sign. $(-1)^{r(x,y)} \mu(x, y) \geq 0$.
- $\mu$ has only 5 possible values $\{-2, -1, 0, 1, 2\}$.
- $CT_n$ admits a symmetric chain decomposition.

We studied $CT_n$ by looking at the quotient of Boolean lattice.
The **Boolean lattice**, denoted $B_n$, is the power set of $[n] = \{1, 2, \ldots, n\}$ ordered by inclusion. An element $A \in B_n$ can be viewed as an $n$-bit binary string whose $i$th bit is 1 if $i \in A$, 0 if $i \not\in A$. 
inversion on $B_n$: $x = (x_1, x_2, \ldots, x_n)$, $x' = (x_n, x_{n-1}, \ldots, x_1)$.

For $x, y \in B_n$, we say $x \sim y$ if $y' = x$.

Then $B_n/\mathbb{Z}_2$ is the quotient poset of $B_n$ under the equivalence relation $\sim$.

0100 < 1010 in $B_4/\mathbb{Z}_2$ because 0010 < 1010 in $B_4$. 
Let $CT_n$ be the poset of caterpillar trees with $n$ vertices induced by KC transformation. There is a bijection between $CT_{n+2}$ and $B_n/\mathbb{Z}_2$. 
Bijection

[Diagram showing a bijection between two sets represented with graphs and number assignments.]
\[
\begin{pmatrix}
1 & -1 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 1 \\
0 & 1 & 0 & -1 & -1 & -1 & 0 & 1 & 2 & -1 \\
0 & 0 & 1 & -1 & -1 & 0 & -1 & 2 & 1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
We classify the orbits in $B_n/\mathbb{Z}_2$ by its elements in $B_n$. 

\begin{align*}
\begin{array}{c}
\text{B}_n \\
\text{A} & y=y' \\
\text{B} & y \quad y' \\
\text{C} & y \quad y' \\
\text{D} & y=y' \\
\text{E} & y \quad y' \\
\text{B}_n/\mathbb{Z}_2 \\
\text{y} \\
\text{x}
\end{array}
\end{align*}
For any $x, y \in B_n/\mathbb{Z}_2$, $x < y$, we have the following formula for its Möbius function.

$$
\mu(x, y) = \begin{cases} 
(-1)^{\ell} & \text{if } (x, y) \in B, D, E \\
2(-1)^{\ell} & \text{if } (x, y) \in C \\
\frac{1}{2}((-1)^{\ell} + (-1)^{\lceil \frac{\ell+1}{2} \rceil}) & \text{if } (x, y) \in A
\end{cases}
$$

where $\ell = |y| - |x|$.

The Möbius function alternates in sign
Symmetric Chain Decomposition on $B_n$
Duffus, Mckibben-Sanders and Thayer (2011) showed the existence of symmetric chain decomposition on $B_n/\mathbb{Z}_2$. 

![Diagram showing symmetric chain decomposition on $B_n/\mathbb{Z}_2$.]
Question (Z. Lin, 2014)

Möbius function of $T_n$? Alternate in sign?
Is $T_n$ EL shellable? Cohen-Macaulay?
References


Thank You!